Coupling and Simulation of Lumped Electric Circuits Refined by 3-D Magnetoquasistatic Conductor Models Using MNA and FIT

Master Thesis

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January 9, 2008
Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher, und insofern sie sicher sind, beziehen sie sich nicht auf die Wirklichkeit.

Albert Einstein, [27]
Preface

This treatise would not have been possible without the work and help of others. I would like to thank Prof. Dr. M. Günther (BU Wuppertal) for introducing me into the topic and entrusting this treatise to me, furthermore Prof. Dr. H. De Gersem (Katholieke Universiteit Leuven) for the inestimably valuable help in electromagnetics and Dr. A. Bartel’s (BU Wuppertal) continuous guidance.

Additional credit goes to Prof. Dr. M. Clemens for his advice regarding theoretical properties of the Finite Integration Technique and to CST, whose software EM Studio was used for the modelling and visualization of the examples.

Finally I like to thank my parents, my girlfriend and all my other friends for giving my encouragement and my former English teacher for proof-reading this treatise.

Wuppertal, December 2007

Sebastian Schöps
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<tr>
<td>$\vec{B}$</td>
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<tr>
<td>$c_0$</td>
<td>velocity of light</td>
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<tr>
<td>$\vec{D}$</td>
<td>electric flux density</td>
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<tr>
<td>$\vec{E}$</td>
<td>electric field strength</td>
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<td>$\epsilon$</td>
<td>electric permittivity</td>
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<td>$\vec{H}$</td>
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<td>$\mu$</td>
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<td>$\rho$</td>
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<td>$\sigma$</td>
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### Field Discretization

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<td>$\vec{j}$</td>
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<td>$K_\nu$ (gauged) curl-curl matrix</td>
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<td>$M_\varepsilon$</td>
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<td>$j_V$</td>
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<td>$\lambda$</td>
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<td>$\phi_L$</td>
<td>magnetic flux through inductors</td>
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$q_C$ charges of conductors ........................................ 34
$u$ vector of branch voltages ........................................ 32
$v$ independent voltage source function .......................... 36
$x_{\text{mna}}$ vector of network variables $x_{\text{mna}} := (e, jL, jV)^T$ .......................... 37

Conductor Models

$\Gamma_{\text{sol}}$ cross section through a solid conductor .............. 41
$\Gamma_{\text{str}}$ cross section through a stranded conductor ............ 41
$\mathbf{i}_{\text{sol}}$ currents through solid conductors .......................... 45
$\mathbf{i}_{\text{str}}$ currents through stranded conductors ................. 47
$\ell_{\text{sol}}$ length of a solid conductor .................................. 41
$\ell_{\text{str}}$ length of a stranded conductor ............................... 41
$M_{\sigma, \text{aniso}}$ anisotropic conductivity matrix .......................... 47
$M_{\sigma, \text{ass}}$ assembled conductivity matrix ............................ 49
$M_{\sigma, \text{fillin}}$ assembled conductivity matrix with fill-ins ............ 58
$M_{\sigma, \text{iso}}$ isotropic conductivity matrix ............................... 43
$Q_{\text{sol}}$ 3-D coupling matrix for solid conductors ................. 46
$Q_{\text{str}}$ 3-D coupling matrix for stranded conductors ............. 48
$u_{\text{sol}}$ voltage drops of solid conductors ............................. 45
$u_{\text{str}}$ voltage drops of stranded conductors .......................... 47
$x_{\text{fit}}$ vector of coupled conductor variables $x_{\text{fit}} := (\bar{a}, \mathbf{i}_{\text{sol}}, \mathbf{i}_{\text{str}})^T$ .................. 50

Coupling

$\lambda$ vector of currents through conductors, composed of $\mathbf{i}_{\text{sol}}$ and $\mathbf{i}_{\text{str}}$ .......................... 62
$u_{\text{fit}}$ vector of conductor voltage drops, composed of $u_{\text{sol}}$ and $u_{\text{str}}$ .......................... 62
LIST OF SYMBOLS
Chapter 1

Introduction

1.1 Motivation

Electric circuits contain devices that exhibit multi-physical effects. We may think of electric or magnetic, but also thermal effects. Traditionally these devices are idealized and only one effect is considered, while the others are disregarded. This yields simple laws that mathematically express their transient behaviour, but does not conform to reality. So sometimes these models are not accurate enough and one wants to simulate a particular device with a refined model. We shall present such a refined modelling approach for electromagnetic devices in this treatise.

There is on one hand the Modified Nodal Analysis (MNA) which allows us to simulate even very large electric circuits. On the other hand there is the Finite Integration Technique (FIT) that is a consistent reformulation of Maxwell’s Equations on discrete grids. It is used today for the efficient and accurate simulation of electromagnetic fields. Both approaches base on the same theory that goes back to James Clerk Maxwell, but have different focuses.

We shall embed both approaches in a consistent theory and establish a coupling that allows the solution of field/circuit problems. Finally we will analyze its properties and show that the coupled problem is well posed and that the simulation is numerically feasible.

Let us now substantiate what we understand under a coupled problem and why our approach is advantageous.


1.2 Coupled Problems

Coupled Problems are characterised by their decomposition into two or more sub-problems. The multiplicity of these sub-problems may arise through refined modelling or multiphysical extension. Their interdependence is stated as either physically weak or strong, depending on the degree of impact which the physical sub-problems have on each other. The problems are numerically tackled by one of the following two approaches

- **monolithic.** All sub-problems are combined into one single system of equations and are solved simultaneously.

- **co-simulation.** The sub-problems are solved sequentially or in parallel. The coupling is restricted to the interchange of partial solutions for particular time frames.

The diversification of co-simulation methods goes further, since the methods and the time-steppings applied to the sub-problems may vary (*multi-method* and *multi-rate*). Furthermore a single time frame may be computed iteratively (*dynamic iteration*) or only once (*full multi-rate co-simulation*).

In our case of field/circuit coupled models we deal with refined modelling. The first sub-problem concerns the electric circuit. It is described by a system of differential algebraic equations (DAE), which results from a network approach to avoid a simulation of the complete circuit using Maxwell’s equations. Only
parts with critical magnetic fields are simulated as a second sub-problem, i.e., the circuit is refined in those parts by a 3-D model. This refinement is described by Maxwell’s set of partial differential algebraic equations (PDAE), but comes as another system of DAEs after the spatial discretization. Fig. 1.1 shows an exemplary lumped circuit that is refined by such a 3-D model. We shall discuss this particular example in Section 6.2.

Field and circuit models are often solved monolithically since most of them are physically strongly coupled. On the other hand the magnetoquasistatic problems that are in focus of this treatise may have parts which are physically weakly coupled in time. Large differences in the time constants can occur, since the propagation in the electric circuit is much faster than in the magnetic field (multi-scale). Therefore a multi-rate method is considered advantageous. Additionally the co-simulation approach is very popular in circuit analysis today because various device models may be refined using external simulators each without the need for interfaces to the others.

1.3 Overview

This treatise is divided into three parts, the first two are devoted to the introduction of both the field and circuit simulations while the main part concerns their coupling.

We start with the theory of field simulation in Chapter 2. The first Section 2.1 on electromagnetism introduces Maxwell’s theory with a short synopsis of the needed physical background, where the focus is on the magnetoquasistatic problem. The next Section 2.2 presents the discretization of Maxwell’s Equations in terms of FIT. We finally discuss the numerical properties of this discretization scheme in the last Section 2.3.

Chapter 3 on circuit simulation is analogously structured as its predecessor. The physical background of the network approach is given in Section 3.1 which is quite compact since it refers to Maxwell’s theory. Afterwards we cover the MNA and their numerical properties in Section 3.2 and Section 3.3 respectively.

In the first section of Chapter 4 two FIT models the solid and stranded conductor are derived. Those models are not MNA specific and can be coupled to any circuit model. We analyze subsequently the numerical properties in the last Section 4.2.

We finally establish a coupling in Chapter 5 and analyze the properties for two different approaches the monolithic and co-simulation. Chapter 5.3 takes up the subject of both approaches again and discusses simple simulation algorithms. Those algorithms are tested on several examples that are presented in Chapter 6.
Finally in the last Chapter 7 of this treatise the results are summarized and we give an outlook on the topics that have not been analyzed in detail, but are surely of interest for further research.

The Appendices concentrate basic results from linear algebra and from the theory of differential equations in A and B respectively. Appendix C includes the headers with brief explanations of important m-files that were used for the MATLAB simulations.
Chapter 2

FIT Field Simulation

2.1 Electromagnetism

All phenomena of electromagnetic fields are described by the set of equations postulated by James Clerk Maxwell and first published in 1865.

In Maxwell’s original formulation 20 equations were grouped into eight systems (A)–(H), all written in component notation, see [44]. In 1892 Oliver Heaviside translated these equations into an equivalent, more compact vectorial formulation (HSE), which reads

\[ \vec{J} = \vec{J}_{\text{cond}} + \frac{\partial \vec{D}}{\partial t}, \]  
(2.1a)

\[ \mu \vec{H} = \nabla \times \vec{A}, \]  
(2.1b)

\[ \nabla \times \vec{H} = \vec{J}, \]  
(2.1c)

\[ \vec{E} = \frac{1}{\varepsilon} \vec{D}, \]  
(2.1d)

\[ \vec{E} = \frac{1}{\sigma} \vec{J}_{\text{cond}}, \]  
(2.1e)

\[ \nabla \cdot \vec{D} = \rho, \]  
(2.1f)

\[ \nabla \cdot \vec{J}_{\text{cond}} = -\frac{\partial \rho}{\partial t}. \]  
(2.1g)

The first relation (2.1a) is Maxwell’s law of total currents. It establishes a relation between the electric displacement current density \( \vec{J}_{\text{disp}} = \frac{\partial \vec{D}}{\partial t} \), current densities due to conduction \( \vec{J}_{\text{cond}} \) and the total current density \( \vec{J} \), compounded of both.
The displacement current density was introduced by Maxwell to ensure the consistence of the conduction current flowing in the wire and the current between two capacitor plates. The second one (2.1b) defines the magnetic vector potential $\vec{A}$ as the origin of the magnetic field strength $\vec{H}$. Ampère’s law (2.1c) describes the relation between the strength of the current density $\vec{J}$ and its magnetic effects $\vec{H}$. The fourth law (2.1d) decomposes the electric field $\vec{E}$ into its components due to convection, induction and charges, where the latter is expressed in terms of the electric scalar potential $\varphi$. The electric elasticity equation (2.1e) relates the electric flux density $\vec{D}$ with the electric field strength $\vec{E}$, which produces it. Similarly to the previous a relation, Ohm’s law (2.1f) establishes a relation between the electric current density $\vec{J}_{\text{cond}}$ and its producing electric field $\vec{E}$. Gauss’ law (2.1g) states the relation between the electric charge density $\rho$ at any point and the electric displacement current in its neighbourhood. Finally the last one (2.1h) guarantees the continuity of charges, in relating the increase or decrease of the electric charge density to the electric current density $\vec{J}_{\text{disp}}$ in the neighbourhood [44]. See Table 2.1 for a quick overview of the involved quantities and their units.

The magnetic flux density $\vec{B}$ was not introduced in the HSE, but it can easily be added with the simple additional definition

$$\vec{H} = \frac{1}{\mu} \vec{B},$$

(2.2)

which reflects a connection of magnetic field strength and magnetic flux density, which is similar to the electric case, cf. Equation (2.1c). This definition allows us to decompose the system in general field describing equations and material dependent relations. We will start with the latter.

### 2.1.1 Material Relations

Heaviside’s system of equations contains several constitutive relations. When inserting the additional equation (2.2) in the HSE, the parameter $\mu$ is entirely removed from the system and only three equations reflect a dependence of material behaviour: (2.2), (2.1e) and (2.1f). The former two include implicitly several universal constants, which define properties of vacuum:

- velocity of light $c_0$,
- permittivity $\varepsilon_0$,
- permeability $\mu_0$.

The velocity of light has been determined experimentally with increasing precision during the last 200 years. Since October 21, 1983, it has been included as a
2.1. ELECTROMAGNETISM

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<td>[V/m]</td>
</tr>
<tr>
<td><strong>$\vec{H}$</strong> magnetic field strength</td>
<td>[A/m]</td>
</tr>
<tr>
<td><strong>$\vec{D}$</strong> electric flux density</td>
<td>[C/m²]=[As/m²]</td>
</tr>
<tr>
<td><strong>$\vec{B}$</strong> magnetic flux density</td>
<td>[T]=[Vs/m²]</td>
</tr>
<tr>
<td><strong>$\vec{J}$</strong> electric current density</td>
<td>[A/m²]</td>
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<tr>
<td>$\rho$ electric charge density</td>
<td>[C/m³]</td>
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<thead>
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<tr>
<td><strong>$\varepsilon$</strong> electric permittivity</td>
<td>[F/m]=[As/Vm]</td>
</tr>
<tr>
<td><strong>$\mu$</strong> magnetic permeability</td>
<td>[H/m]=[Vs/Am]</td>
</tr>
<tr>
<td><strong>$\sigma$</strong> electric conductivity</td>
<td>[S/m]=[A/Vm]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Potential quantities</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$\vec{A}$</strong> magnetic vector potential</td>
<td>[Vs/m]</td>
</tr>
<tr>
<td><strong>$\varphi$</strong> electric potential field</td>
<td>[V]</td>
</tr>
</tbody>
</table>

Table 2.1: Electromagnetic Quantities. Overview of symbols, descriptions and units in Maxwell’s equations, a similar can be found in [8].

The definition in the Système International d’Unités (SI). Today the metre is the derived quantity. It is the length of the path travelled by light in a vacuum during a time interval of 1/299 792 458 of a second [45]. The definition of the velocity of light reads

$$c_0 = 2.99792458 \cdot 10^8 \text{ m/s}.$$ (2.3)

The constants $\varepsilon_0$ and $\mu_0$ are proportionality factors in the vacuum between the electric flux density $\vec{D}$ and the electric field strength $\vec{E}$, and between the magnetic flux density $\vec{B}$ and the magnetic field strength $\vec{H}$, respectively:

$$\vec{D} = \varepsilon_0 \vec{E},$$ (2.4)
$$\vec{B} = \mu_0 \vec{H}.$$ (2.5)

The permeability $\mu_0$ of the free space is independent of the velocity of light $c_0$ and is chosen in SI units as

$$\mu_0 = 4\pi \cdot 10^{-7} \text{ H/m}.$$ (2.6)

Now that the permeability $\mu_0$ is fixed, the velocity of light $c_0$ defines the permittivity of free space in SI units to

$$\varepsilon_0 = 8.854 \cdot 10^{-12} \text{ F/m},$$ (2.7)
using the relation
\[ c_0 = \frac{1}{\sqrt{\varepsilon_0\mu_0}}. \tag{2.8} \]
This relation was postulated by Maxwell together with the fact that light itself is an electromagnetic wave \[43\].

The simple equations (2.4) and (2.5) for the free space extend to more general material relations
\[ \vec{D} = \varepsilon_0 \vec{E} + \vec{P}, \tag{2.9} \]
\[ \vec{B} = \mu_0 (\vec{H} + \vec{M}), \tag{2.10} \]
with the electric polarization \(\vec{P}\) and the magnetization \(\vec{M}\), which are valid for any material. The equations simplify again, when taking just linear, homogeneous, isotropic and time-invariant materials into account. In this case the polarization and magnetization are set in relation to the electric field strength and the magnetic field strength respectively
\[ \vec{P} = \varepsilon_0 \chi_e \vec{E}, \tag{2.11} \]
\[ \vec{M} = \mu_0 \chi_m \vec{H}, \tag{2.12} \]
where \(\chi_e\) is the dielectric susceptibility and \(\chi_m\) is the magnetic susceptibility. If we define the relative permittivity of a dielectric material by \(\varepsilon_r := (1 + \chi_e)\) and the relative permeability of a magnetic material by \(\mu_r := (1 + \chi_m)\), the behaviour of a linear, homogeneous and isotropic material simplifies to
\[ \vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}, \tag{2.13} \]
\[ \vec{B} = \mu_0 \mu_r \vec{H} = \mu \vec{H}, \tag{2.14} \]
with the augmented material parameters \(\varepsilon\) and \(\mu\) that are constant and independent of the fields \(\vec{E}\) and \(\vec{H}\) and any other influences, e.g. time or directions. We will concentrate on this case in this treatise, because it simplifies our analysis.

The third and last constitutive relation (2.11) is Ohm’s law, which is well-known in its macroscopic version. The continuum form describes a linear dependence between the conduction current density \(\vec{J}_{\text{cond}}\) and the electrical field strength \(\vec{E}\) with a proportionality factor, which is called the conductivity \(\sigma\):
\[ \vec{J}_{\text{cond}} = \sigma \vec{E}. \tag{2.15} \]

2.1.2 Maxwell’s Equations

When we speak about Maxwell’s equations (ME) today, we mean neither the original component-wise set, nor Heaviside’s above (2.1), but the following reduced
system in its differential or integral form \[56\]. The magnetic vector potential \( \vec{A} \) and the electric scalar potential \( \varphi \) are eliminated. The material relations and the continuity equation (2.11h) are separated.

The reduced set reads in differential form:

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad (2.16a) \]
\[ \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}, \quad (2.16b) \]
\[ \nabla \cdot \vec{D} = \rho, \quad (2.16c) \]
\[ \nabla \cdot \vec{B} = 0. \quad (2.16d) \]

Applying Stoke’s Theorem to (2.16a) and (2.16b) as well as Gauß’ Theorem to the remaining equations (2.16c) and (2.16d) yields the system in integral form:

\[ \int_{\partial A} \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \int_{A} \vec{B} \cdot d\vec{A}, \quad (2.17a) \]
\[ \int_{\partial A} \vec{H} \cdot d\vec{s} = \int_{A} \left( \frac{\partial \vec{D}}{\partial t} + \vec{J} \right) \cdot d\vec{A}, \quad (2.17b) \]
\[ \int_{\partial V} \vec{D} \cdot d\vec{A} = \int_{V} \rho \cdot d\vec{V}, \quad (2.17c) \]
\[ \int_{\partial V} \vec{B} \cdot d\vec{A} = 0. \quad (2.17d) \]

Although Maxwell earns the credit for grouping all the equations together into a coherent set and introducing the displacement current, he incorporated the work of several well-known physicians. For this reason the individual equations are still attributed to earlier scientists. In consistency with HSE, the first equation \( (2.16a) \) is known as Faraday’s law, the second one \( (2.16b) \) is named after Ampère and \( (2.16d) \) is called Gauß’ law.

The last equation \( (2.16d) \) was not needed in HSE, because it is a direct result of the existence of the magnetic vector potential \( \vec{A} \), see Chapter 2.1.4. In this notation it is indicated that magnetic monopoles do not exist.

### Electric Current Density

An augmented current density \( \vec{J} \) was already defined in HSE as the sum of current densities due to displacement and conduction. We generalize this by taking more densities into account which correspond to particular electromagnetic effects

\[ \vec{J} = \vec{J}_{\text{cond}} + \vec{J}_{\text{src}} + \vec{J}_{\text{conv}} + \vec{J}_{\text{disp}}. \quad (2.18) \]
Now the current densities involved are

**conduction current density** $\vec{J}_{\text{cond}}$, which is a result of the electric field $\vec{E}$ and the friction due to collisions of the flowing electrons with the static atoms of the conducting material. It is macroscopically defined by Ohm’s law, see Equation (2.1f) of HSE,

**source current density** $\vec{J}_{\text{src}}$, which describes an external excitation that is independent of all other forces. The introduction of $\vec{J}_{\text{src}}$ is a generalization of the HSE case.

**convection current density** $\vec{J}_{\text{conv}} = \rho \vec{v}$, resulting from a charge density $\rho$, which moves with speed $\vec{v}$. This quantity is introduced here just for completeness; it was neither included in HSE, nor it will be regarded in our models.

**displacement current density**

$$\vec{J}_{\text{disp}} = \frac{\partial \vec{D}}{\partial t},$$

(2.19)

which is explicitly mentioned in Ampère’s law. This effect will be disregarded in this treatise because of the magnetoquasistatic assumption that is introduced in the following section.

### 2.1.3 Magnetoquasistatic Approximation

There are several approaches to simplifying the calculations of the electromagnetic field, each disregarding different effects, see Fig. 2.1 for an overview. It is obvious that such solutions are just approximations to the “exact” ones, which could be obtained from solving the full set of Maxwell’s equations, but there are error estimates and revisable conditions to validate the particular simplifications in each case.

We concentrate on the magnetoquasistatic (MQS) approximation. It means that the fields considered are so slowly varying with no spatial propagation that they can be regarded as quasistationary or static. This holds true for example if the wavelength $\lambda$, which depends on the frequency $f$, defined by

$$\lambda = \frac{1}{f \sqrt{\varepsilon \mu}}$$

is much larger than four times the typical length scale $\ell$ of the area considered, or, as written in [56]

$$\ell \ll \frac{\lambda}{4}.$$
This particular approach is feasible if the contribution of the displacement currents can be disregarded with respect to the conduction currents, in other words

$$\max \left| \frac{\partial \vec{D}}{\partial t} \right| \ll \max \left| \vec{J}_{\text{cond}} \right|.$$  (2.20)

Applying this approximation to System (2.16) leads to:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$  (2.21a)

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_{\text{cond}},$$  (2.21b)

$$\nabla \cdot \vec{D} = \rho,$$  (2.21c)

$$\nabla \cdot \vec{B} = 0.$$  (2.21d)

Now equations (2.21b) and (2.21d) describe together with (2.14) the magnetic field uniquely. For a general overview of magnetoquasistatic formulations see Haus and Melcher [35] and Dirks [25].

### 2.1.4 Electrodynamic Potentials

In their original formulation Maxwell and Heaviside used more variables than were actually needed. The system of equations contained additionally the magnetic...
vector potential (MVP) $\vec{A}$ and the electric scalar potential $\varphi$, and even more potentials like the magnetic scalar potential $\psi$ could have been introduced.

We need the former two potentials in this treatise in order to simplify our discretization scheme later on. Therefore we split the electric field into rotational and irrotational parts. The simplest case of Poincaré’s lemma guarantees the representation of irrotational vector fields as gradient fields of scalar potentials on a simply connected domain, i.e.,

$$\vec{B} = \nabla \times \vec{A},$$  \hspace{1cm} (2.22) \\
$$\vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t}.$$  \hspace{1cm} (2.23)

The expression of (2.23) in terms of both potentials is called the $(\vec{A}, \varphi - \vec{A})$-formulation. It is important to note that the potentials $\vec{A}$ and $\varphi$ are not unique, while the magnetic flux density $\vec{B}$ and the electric field strength $\vec{E}$ are necessarily unique, and they retain unvaried by the following gauge transformation [25]

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla \xi,$$  \hspace{1cm} (2.24) \\
$$\varphi \rightarrow \varphi' = \varphi - \frac{\partial \xi}{\partial t}.$$  \hspace{1cm} (2.25)

We easily verify the coherence using the definitions of the magnetic and electric flux. It holds

$$\vec{B}' = \nabla \times \vec{A}'$$  \\
$$= \nabla \times (\vec{A} + \nabla \xi)$$  \\
$$= \nabla \times \vec{A} + (\nabla \times \nabla \xi)$$  \\
$$= \vec{B},$$

and

$$\vec{E}' = -\nabla \varphi' - \frac{\partial \vec{A}'}{\partial t}$$  \\
$$= -\nabla \left( \varphi - \frac{\partial \xi}{\partial t} \right) - \frac{\partial (\vec{A} + \nabla \xi)}{\partial t}$$  \\
$$= -\nabla \varphi - \frac{\partial \vec{A}}{\partial t} + \frac{\partial \nabla \xi}{\partial t} - \frac{\partial \nabla \xi}{\partial t}$$  \\
$$= \vec{E},$$

which is similar to the derivations in [56].
2.1.5 Magnetoquasistatic Formulation in Terms of MVP

Using the potentials of the previous chapter we can rewrite Maxwell’s system to give a more compact formulation. As a consequence, the set of four first order differential equations simplifies to two second-order differential equations, see [11, 10]. In the following, such a formulation will be derived for the magnetoquasistatic case. The resulting system of equations has the nature of a diffusion equation, since wave propagation effects are neglected. For a general approach see Dirks [25].

The introduction of the magnetic material relation \( \vec{H} = \mu^{-1} \vec{B} =: \nu \vec{B} \) and Ohm’s law \( J_{\text{cond}} = \sigma \vec{E} \) in Ampere’s law (2.1c) without displacement currents leads to

\[
\nabla \times (\nu \vec{B}) = J_{\text{src}} + \sigma \vec{E}.
\]

Introducing the potentials instead of \( \vec{E} \) and \( \vec{B} \) leads to the continuous MQS curl-curl equation in \( \vec{A} \)-\( \varphi \)-formulation

\[
\nabla \times (\nu \nabla \times \vec{A}) + \sigma \frac{\partial \vec{A}}{\partial t} + \sigma \nabla \varphi = J_{\text{src}}.
\]

The remarks in the previous section about the uniqueness of the potentials still apply here and this freedom can be exploited to choose \( \nabla \varphi = 0 \), which yields a special of equation (2.27), the so-called \( \vec{A}^\star \)-formulation

\[
\nabla \times (\nu \nabla \times \vec{A}) + \sigma \frac{\partial \vec{A}}{\partial t} = J_{\text{src}}.
\]

Another equation can be obtained from (2.27) by applying the divergence operators to both sides of (2.27), yielding (2.29), since the right hand side is solenoidal.

\[
-\nabla \cdot \left( \sigma \frac{\partial \vec{A}}{\partial t} \right) - \nabla \cdot (\sigma \nabla \varphi) = 0.
\]

2.2 Finite Integration Technique

2.2.1 Maxwell’s Grid Equations

The Finite Integration Technique (FIT) is a spatial discretization scheme for solving Maxwell’s equations. It translates the continuous Maxwell equations (ME) one by one into a space-discrete set, called the Maxwell grid equations (MGE). The method was developed by Weiland [59] in 1977 and is, when formulated
Electromagnetic field problems are in general an open boundary problem. However we need to restrict the computational domain. In FIT only a simple connected and bounded space region \( \Omega \in \mathbb{R}^3 \) is considered. In the next step its topology has to be approximated by a finite number of cells \( V(n) \) for \( 1 \leq n \leq N \). These cells are hexahedra in the case of a Cartesian mesh as explained here, but could be of any other type, such as that arising from cylindrical or spherical coordinates, triangular or even unstructured grids, see for example [50].

We focus on the Cartesian case as given in [16]. The construction yields the finite cell complex \( G \), composed of intervals defined by equidistant distributed coordinates \( x_i, y_j \) and \( z_k \)

\[
G := \{ V(n) := V(i, j, k) \mid V(i, j, k) = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}] ;
\]

\[
i = 1, \ldots, I - 1; j = 1, \ldots, J - 1; k = 1, \ldots, K - 1 \},
\]

where the three indices \( i, j \) and \( k \) are combined into one space index, which allows us to number the elements consecutively:

\[
n = n(i, j, k) = i + (j - 1) \cdot I + (k - 1) \cdot I \cdot J ,
\]

such that

\[
n \leq N := I \cdot J \cdot K .
\]

The intersection of two volumes is by construction either empty for non-neighboring volumes or one of the following \( p \)-cells, where \( p \in \{0, 1, 2, 3\} \) denotes the dimension of the geometrical object and \( w \in \{x, y, z\} \) a direction in space.
2.2. **FINITE INTEGRATION TECHNIQUE**

- 0-cell: a simple point \( P(n) \),
- 1-cell: an edge \( L_w(n) \) of length \( \Delta w = w_{i+1} - w_i \),
- 2-cell: a rectangle, called facet \( A_w(n) \),
- 3-cell: a volume \( V(n) \).

Every object is associated with its smallest numbered connected point \( P(n) \). An edge \( L_w(n) \) connects two in \( w \)-direction neighboured points \( P(n) \) and \( P(n') \) \( (n < n') \) and is always directed from \( P(n) \) towards \( P(n') \). A facet \( A_w(n) \) is defined by \( P(n) \) and the direction \( w \), in which its normal vector points, see Fig. 2.2(a).

The basic idea of FIT is the usage of two grids, in a similar way as for the well-known leap-frog scheme. The primary grid \( G \) is supported by the dual grid \( \tilde{G} \), which is identically but shifted in \( x-, y- \) and \( z \)-direction by half of a cell length, see Fig. 2.2(b). The definition of the dual \( p \)-cells, i.e., edges \( \tilde{L}_w(n) \), facets \( \tilde{A}_w(n) \) and volumes \( \tilde{V}(n) \) is analogous to the primary grid \( (w \in \{x, y, z\}) \). In the following each primary \( p \)-cell of \( G \) will be related to one \((3 - p)\)-cell of \( \tilde{G} \).

As state variables of the FIT, we introduce electric and magnetic voltages and fluxes. They are defined as integrals of the electric and magnetic field strengths and flux densities over geometrical objects of the computational grid, with respect to the directions \( w \in \{x, y, z\} \). The state variables are assigned diacritics (\( \tilde{\cdot} \)) according to their dimension \( p \) of the underlying object. This topic will be discussed in more detail in Section 2.2.3. The grid voltages over the edges read

\[
\vec{e}_w(n) = \int_{L_w(n)} \vec{E} \cdot d\vec{s},
\]

\[
\tilde{\vec{h}}_w(n) = \int_{\tilde{L}_w(n)} \tilde{\vec{H}} \cdot d\vec{s}.
\]

The fluxes are located on the grid facets and read

\[
\tilde{\vec{b}}_w(n) = \int_{\tilde{A}_w(n)} \tilde{\vec{B}} \cdot d\tilde{\vec{A}},
\]

\[
\tilde{\vec{d}}_w(n) = \int_{\tilde{A}_w(n)} \tilde{\vec{D}} \cdot d\tilde{\vec{A}},
\]

\[
\tilde{\vec{j}}_w(n) = \int_{\tilde{A}_w(n)} \tilde{\vec{J}} \cdot d\tilde{\vec{A}}.
\]

To simplify the notation we will build augmented vectors for each of the newly defined quantities with a length of \( N_{\text{fit}} = 3 \cdot N \), including every spatial direction. For example the discrete electric field strengths are collected in

\[
\vec{e} = (\vec{e}_x(1), \ldots, \vec{e}_x(N), \vec{e}_y(1), \ldots, \vec{e}_y(N), \vec{e}_z(1), \ldots, \vec{e}_z(N))^T.
\]
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Figure 2.3: Maxwell’s Grid Equations. Two examples of the MGE derived by FIT using a single grid cell; similar figures are found in [8, 16, 20]

The remaining vectors \( \vec{h}, \vec{b}, \vec{d} \) and \( \vec{j} \) are defined analogously. Using these notations we are able to rewrite ME (2.17) in terms of the FIT discretization. For example, Faraday’s law (see Fig. 2.3(a)) for a single grid facet \( A_z(n) \) can be rewritten as

\[
\vec{e}_x(n) + \vec{e}_y(n + 1) - \vec{e}_x(n + I) - \vec{e}_y(n) = - \frac{d}{dt} \vec{b}_z(n), \tag{2.39}
\]

which exploits the new order of numbering and is easily generalized to all facets. The relations for all grid facets are collected in the matrix equation:

\[
\begin{pmatrix}
\cdots & 1 & \cdots & -1 & \cdots & -1 & 1 & \cdots \\
\vdots & & & & & & & \\
\vec{c} & & & & & & & \\
\end{pmatrix}
\begin{pmatrix}
\vec{e}_x(n) \\
\vec{e}_x(n + I) \\
\vec{e}_y(n) \\
\vec{e}_y(n + 1) \\
\vdots \\
\vec{e} \\
\end{pmatrix}
= - \frac{d}{dt} \begin{pmatrix}
\vec{b}_z(n) \\
\vdots \\
\vec{b} \\
\end{pmatrix}. \tag{2.40}
\]

The law of the non-existence of magnetic monopoles can be treated analogously, see Fig. 2.3(b). We derive for the single grid cell \( V_n \)

\[
-\vec{b}_x(n) + \vec{b}_x(n + 1) + \vec{b}_y(n) + \vec{b}_y(n + I) - \vec{b}_z(n) + \vec{b}_z(n + I.J) = 0. \tag{2.41}
\]
2.2. FINITE INTEGRATION TECHNIQUE

When we assemble the coefficients in a matrix, we get

\[
\begin{pmatrix}
... & -1 & 1 & ... & -1 & 1 & ... & -1 & 1 & ... \\
\vdots & & & & & & & & & \\
\end{pmatrix}
\]

\[= S\]

\[
\begin{pmatrix}
\tilde{b}_x(n) \\
\tilde{b}_x(n+1) \\
\vdots \\
\tilde{b}_y(n) \\
\vdots \\
\tilde{b}_y(n+I) \\
\vdots \\
\tilde{b}_z(n) \\
\vdots \\
\tilde{b}_z(n+IJ) \\
\vdots \\
\end{pmatrix}
\]

\[= 0. \quad (2.42)\]

Repeating this procedure for the remaining two equations of ME yields two additional equations with matrices \(\tilde{C}\) and \(\tilde{S}\) that live on the dual grid. All equations together give Maxwell’s Grid Equations (MGE). It resembles remarkably closely the continuous system \((2.16)\) and maintains several of its properties \([51]\).

\[
C\tilde{\varepsilon} = -\frac{d}{dt} \tilde{\varepsilon} \tilde{b}, \quad (2.43a)
\]

\[
\tilde{C}\tilde{\mu} = \frac{d}{dt} \tilde{\mu} + \tilde{j}_{\text{cond}}, \quad (2.43b)
\]

\[
\tilde{S}\tilde{d} = q, \quad (2.43c)
\]

\[
\tilde{S}\tilde{b} = 0. \quad (2.43d)
\]

The \(n\)-dimensional vector \(q\) contains the electrical charges allocated at the dual grid cells, in analogy to \(\rho\) in the continuous case. The \(N_{\text{fit}} \times N_{\text{fit}}\)-matrices \(C\) and \(\tilde{C}\) pick out the components of the corresponding vectors involved. They are simple incidence matrices containing information on the edges and on their orientation, representing the discrete curl operators on the grid. \(S\) and \(\tilde{S}\) contain relations applied to each primary or dual cell respectively and represent the discrete divergence operators at the primary and dual grid. Their dimension is \(N \times N_{\text{fit}}\).

Let us consider the irrotational electric field. The grid voltages are located on cell edges in our model. This suggests an interpretation as a voltage difference
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("drop") of two electric potentials, which are defined on the interconnected grid points. This substantiates for the edge \( L_x(n) \), which connects \( P(n) \) and \( P(n+1) \) to

\[
\bar{e}_x(n) = -\phi(n + 1) + \phi(n).
\]

(2.44)

In accordance with the previous procedure we assemble these quantities in a vector and a matrix

\[
\bar{e} = -G\phi,
\]

(2.45)

with the discrete electric scalar potentials \( \phi = (\phi(1), \ldots, \phi(N)) \) and an incidence matrix \( G \) of dimension \( N_{\text{fit}} \times N \), which resembles the discrete gradient operator at the primary grid.

Finally the FIT method has translated the four Maxwell equations exactly into discrete ones. The only discretization error occurs in the originally unbound space region, which has been simplified to the finite computational domain \( \Omega \). Further errors may arise through deficiencies of the mesh selected \[16\].

2.2.2 Discretization of the Material Relations

Working towards a complete discretized system of ME, the three material relations (2.13), (2.14) and (2.15) have to be related to the discrete quantities allocated at the computational grids. Therefore the fluxes on the primary grid \( G \) have to be related to the voltages on the dual analogon \( \tilde{G} \) and vice versa. Hence, the material relations establish a coupling between both grids. This coupling brings in approximations through averaging processes. In the following, this is exemplarily explained in the simplest of the three cases, see Fig. 2.4. A more detailed derivation of all constitutive relations and their averaged parameters is given by Benderskaya \[8\].

The magnetic flux density \( \vec{B} \) is related to the magnetic field strength \( \vec{H} \) through the permeability \( \mu \) defined in equation (2.14). In coherence with our earlier requirements we will assume that there are local permeabilities \( \mu(n) \) for each grid volume \( V(n) \). When we start with the definition of the discrete magnetic field strength in conjunction with constitutive relation (2.14) and averaging its value over the facet \( A_w(n) \) to \( |\vec{B}| \), we get the integral quantity

\[
\bar{h}_w(n) = \int_{L_w(n)} \vec{H} \cdot d\vec{s} = \int_{L_w(n)} \mu^{-1} \vec{B} \cdot d\vec{s}
\]

(2.46)

\[
= \bar{\mu}^{-1}(n) |\tilde{L}_w(n)| |\vec{B}| + \mathcal{O}(h^1)
\]

(2.47)
Figure 2.4: *Discrete Material Relations.* Deduction of the relation between the discrete magnetic flux density and the discrete magnetic field strength; this figure is taken from [20] with averaged permeabilities $\bar{\mu}(n)$ that gives an error, whose exponent depends on the used discretization grid (in our case of a Cartesian grid $l$ equals 2) and the maximum length of the cell edges $h := \max_{w \in \{x, y, z\}} (L_w(n))$.

In a similar manner, we derive for the magnetic flux density

$$\vec{b}_w(n) = \int_{A_w(n)} \vec{B} \cdot d\vec{A}$$

$$= |A_w(n)| \cdot |\vec{B}| + O(h^{l+1})$$

$$\equiv |A_w(n)| \cdot |\vec{B}|.$$  \hspace{1cm} (2.48)

Equations (2.47) and (2.48) contain the averaged magnetic flux density $|\vec{B}|$, which is unknown. Eliminating this unknown through inserting one equation into the other leads to

$$\vec{b}_w(n) = \bar{\mu}(n) \frac{|A_w(n)|}{|L_w(n)|} \cdot \vec{h}_w(n).$$

Finally when we put the permeabilities into the matrix

$$M_\mu := \text{diag} (\bar{\mu}_x(1), \ldots, \bar{\mu}_x(N), \bar{\mu}_y(1), \ldots, \bar{\mu}_y(N), \bar{\mu}_z(1), \ldots, \bar{\mu}_z(N)),$$

we have derived the discrete material relation

$$\vec{b} = M_\mu \vec{h}.$$ \hspace{1cm} (2.49)
The remaining relations are analogously deducible and read
\[
\begin{align*}
\tilde{d} &= M_\varepsilon \tilde{e}, \\
\tilde{j}_{\text{cond}} &= M_\sigma \tilde{e},
\end{align*}
\]
where the matrices \(M_\varepsilon\) and \(M_\sigma\) represent the discrete permittivities and conductivities respectively.

All discrete material matrices are diagonal in the isotropic case and of dimension \(N_{\text{fit}} \times N_{\text{fit}}\). The former two \(M_\mu\) and \(M_\varepsilon\) are regular, while \(M_\sigma\) is singular in general, because non-conducting regions of the computational domain lead to zero entries on the diagonal of the matrix \[7\].

### 2.2.3 Algebraic Properties of the Matrix Operators

The matrix operators have several interesting properties, each according to properties of vector analysis \[16\]. These properties arise naturally through the simple structure of the operators. They are composed of even simpler matrices, which for their part can be interpreted as discretized partial-differential operators:

\[
C = \begin{pmatrix} 0 & -P_z & P_y \\ P_z & 0 & -P_x \\ -P_y & P_x & 0 \end{pmatrix}, \quad (2.52)
\]

\[
S = (P_x P_y P_z), \quad (2.53)
\]

with \(n \times n\)-matrices

\[
(P_x)_{pq} = \begin{cases} 
-1 & \text{for } p = q \text{ and } i \leq I - 1, \\
+1 & \text{for } p = q + 1 \text{ and } i \leq I - 1, \\
0 & \text{else}; 
\end{cases} \quad (2.54)
\]

\[
(P_y)_{pq} = \begin{cases} 
-1 & \text{for } p = q \text{ and } j \leq J - 1, \\
+1 & \text{for } p = q + 1 \text{ and } j \leq J - 1, \\
0 & \text{else}; 
\end{cases} \quad (2.55)
\]

\[
(P_z)_{pq} = \begin{cases} 
-1 & \text{for } p = q \text{ and } k \leq K - 1, \\
+1 & \text{for } p = q + 1 \text{ and } k \leq K - 1, \\
0 & \text{else}; 
\end{cases} \quad (2.56)
\]

and with row index \(1 \leq p = p(i, j, k) \leq N\) and column index \(1 \leq q \leq N\).

Finally, the discrete gradient matrix is as simple as the others

\[
G = -\tilde{S}^T. \quad (2.57)
\]
2.2. **FINITE INTEGRATION TECHNIQUE**

As stated previously, the discrete curl and divergence matrices inherit key properties from their continuous counterparts. The vector analytical equation

$$\nabla \cdot (\nabla \times \vec{v}) = 0, \quad \forall \vec{v} : \mathbb{R}^3 \rightarrow \mathbb{R}^3. \quad (2.58)$$

is reflected in the matrix equations

$$SC = 0, \quad (2.59)$$
$$\tilde{S}\tilde{C} = 0. \quad (2.60)$$

for our grid doublet \(\{G, \tilde{G}\}\), which is easily shown, because

$$SC = (P_y P_z - P_z P_y, P_z P_x - P_x P_z, P_x P_y - P_y P_x) \quad (2.61)$$

holds and the proof is completed by showing

$$P_u P_v = P_v P_u, \quad u, v \in x, y, z; \quad u \neq v, \quad (2.62)$$

which is only the interchange of partial differential operators, see Bartsch et al. [6].

The duality yields to another simple relation for the discrete curl-matrices

$$C = \tilde{C}^T. \quad (2.63)$$

Now exploiting the identity (2.63) in combination with the transposition of (2.59) and (2.60) results in

$$\tilde{C}\tilde{S}^T = 0, \quad (2.64)$$
$$C\tilde{S}^T = 0. \quad (2.65)$$

which reflect another vector analytical property of the curl and gradient operators

$$\nabla \times (\nabla f) = 0, \quad \forall f : \mathbb{R}^3 \rightarrow \mathbb{R}. \quad (2.66)$$

**Discrete Hodge Operators**

The correspondence to abstract mathematics goes much further: the whole theory of differential forms from vector analysis is applicable. The vectors derived are interpretable as discrete \(p\)-forms, which are mapped with a discrete hodge operator to discrete \((3 - p)\)-forms for \(p \in \{0, 1, 2, 3\}\) in the \(\mathbb{R}^3\), see [36]. The discrete hodge operator \(C\), for example, relates the edges (1-form) of the primary grid to facets (2-form) of the dual grid. This is the more detailed justification for the bow notation of the state variables: 1-forms are assigned the diacritic bow “\(\组建\)” and 2-forms two bows “\(\组建\)” as it now becomes clear. Fig. 2.5 gives an overview of the forms and operators in the continuous case. This approach allows interesting interpretations of the Finite Integration Method, see Auchmann and Kurz [3].
2.2.4 Discrete Potential Formulation for MQS Fields

A discrete potential formulation analogous to the result of Chapter 2.1.5 can be derived in terms of the FIT. The magnetoquasistatic assumption introduced in Section 2.1.3 simplifies the discretized law of Ampère once again to

\[ \tilde{C} \tilde{h} = \tilde{J}_{\text{cond}} \]

and with the material relations (2.51) and (2.49) follows

\[ \tilde{C} M_\nu \tilde{b} = M_\sigma \tilde{e} , \]

with the inverse of the discrete permeability matrix \( M_\nu := M_\mu^{-1} \).

The definition of the continuous magnetic vector potential (2.22) from Section 2.1.4 transfers to

\[ C \tilde{a} = \tilde{b} \]  \hspace{1cm} (2.67)

in the discrete FIT notation, which has no unique solution either. Inserting (2.67) in the discrete version of Faraday’s law (2.43a) gives the discrete \( \tilde{A} - \varphi \)-formulation

\[ \tilde{e} = -\frac{d}{dt} \tilde{a} - G\phi , \]  \hspace{1cm} (2.68)
2.2. FINITE INTEGRATION TECHNIQUE

Figure 2.6: Band Structure of the Conductivity and Curl-Curl Matrix. MATLAB’s spy-plot of the matrices from the transformer example (Section 6.1) with a conducting iron core discretized using a coarse grid

with the discrete electric scalar potential $\phi$. Insertion yields the first order system

$$\tilde{C} M_\nu \tilde{a} + M_\sigma \frac{d}{dt}\tilde{a} + M_\sigma G \phi = \tilde{j}_{src}.$$  \hspace{1cm} (2.69)

In coherence to the continuous case the $\tilde{A}^*$-formulation of equation (2.68) is obtained, when $\phi$ is fixed to zero

$$\tilde{C} M_\nu \tilde{a} + M_\sigma \frac{d}{dt}\tilde{a} = \tilde{j}_{src}.$$  \hspace{1cm} (2.70)

One derives another equation when applying the discrete divergence operator $\tilde{S}$ on both sides of (2.69). It follows with property (2.60)

$$-\tilde{S} M_\sigma \frac{d}{dt}\tilde{a} - \tilde{S} M_\sigma G \phi = 0.$$  \hspace{1cm} (2.71)

The various field-related systems in the following chapters will include the $\tilde{A}^*$-formulation (2.70) which is a differential algebraic equation (DAE) due to the singularity of the mass matrix $M_\sigma$. The stiffness matrix $\tilde{C} M_\nu C$ corresponds the double application of the curl operator and is therefore called curl-curl-matrix. This leads also to the name discrete MQS curl-curl equation for (2.70). Note that the DAE cannot be solved uniquely because of the arbitrariness involved in the definition of the magnetic vector potential (see Section 2.3). Fig. 2.6 shows exemplary spy plots of both matrices to illustrate their band structure and rank deficiencies.
2.2.5 Boundary Conditions

As explained in Section 2.2.1, we deal with artificially bounded problems, which are originally not restricted. To make our model coherent, we need to define boundary conditions (BC) that specify the behaviour of the involved quantities at the geometrical borders of our model.

We concentrate on only one simple condition in this treatise: The so-called *electric BC*, often also described as “flux wall” or “current gate”, see Benderskaya [8]. They lead to a model with perfectly conducting borders, i.e., with infinite electric conductance. This is done by defining all tangent components of the electric field strength to vanish at the boundaries

\[ \vec{E} \times \vec{a}_n = 0. \]  

(2.72)

In consequence the normal components of the magnetic flux density are eliminated \( \vec{B} \cdot \vec{a}_n = 0 \) (“flux wall”), while the normal components of the current density are infinitesimally small \( \vec{J} \cdot \vec{a}_n \approx 0 \) (“current gate”). In MVP formulation this is equivalent to Dirichlet BC and

\[ \vec{A} \times \vec{a}_n = 0 \]  

(2.73)

holds which is easily applicable in (MGE): only tangent components of the discrete magnetic vector potential \( \vec{a} \) have to be zeroed at the boundaries. Information on more sophisticated BC is found in De Gersem et al. [23].

2.3 Numerical Analysis

We shall investigate in the following the numerical properties of the discrete MQS curl-curl equation (2.70) in the \( A^\star \)-formulation, which reads

\[ M_\nu \frac{d}{dt} \vec{a} + \tilde{C} M_\nu C \vec{a} = \vec{j}_{src}, \]

this is obviously a implicit system of linear differential (and algebraic) equations with constant coefficients, to make it short it is of the type (B.7). For a deeper insight we shall take a closer look at the algebraic properties of the involved matrices.

2.3.1 Algebraic Properties

Let us analyze the definiteness of \( \tilde{C} M_\nu C \), which we will refer to as the *curl-curl matrix*. We know its middle matrix \( M_\nu \) is diagonal and positive definite and thus
obviously symmetric, since it is the inverse of the discrete permeability matrix 
$M_\mu$, which was diagonal and positive definite (see Proposition A.1). We define
the square root of $M_\mu$ by taking the square root of all the entries on the diagonal,
which yields the product

$$M_\mu = M^{1/2}_\mu \cdot M^{1/2}_\nu. \quad (2.74)$$

Therefore holds in conjunction with the property $\tilde{C} = C^T$

$$\tilde{C} M_\nu C = (M^{1/2}_\nu C)^T \cdot (M^{1/2}_\nu C), \quad (2.75)$$

which proves the positive semi-positive definiteness and symmetry of the curl-curl
matrix according to Proposition A.2, see [16].

The discrete matrix of conductivities $M_\sigma$ is diagonal and positive semi-definite.
The latter is a physical consequence of the fact that only two kinds of equations
are considered in the discretization process. On one hand there are equations
originating from at least one conducting region. These lead to non-zero terms
of the diagonal of $M_\sigma$, while on the other hand non-conducting regions give null
lines [46].

### 2.3.2 DAE Index

The positive semi-definiteness of $M_\sigma$ yields obviously a Jordan decomposition
into a diagonal block and a block of nilpotency 1, see Nicolet and Delincé [46].
Therefore the algebraic DAE index equals 1 according to Definition B.5, if we
assume the regularity of the matrix pencil. This regularity is not trivial, as we
will show in the following.

**Matrix Pencil**

The arbitrariness that is involved in the definition of the MVP directly implies
that the discrete MQS curl-curl equation (2.70) is not uniquely solvable and hence
cannot have a regular pencil. On the other hand the existence of infinitely many
solutions due to the curl-curl matrix is not a physical problem. Every solution
$\hat{a}$ fulfills our needs, because we are only interested in the unique magnetic field
strength $b = C\hat{a}$. Hence mathematically speaking all MVPs $\hat{a}$ are equivalent
according to the following definition, see Kettunen et al. [39].

**Definition 2.1** (Equivalence Relation). Let the domain of the operator $C$ be
denoted by $\text{dom}(C)$. We say that $\hat{a}^1, \hat{a}^2 \in \text{dom}(C)$ are equivalent if
$C\hat{a}^1 = C\hat{a}^2$, and we write $\hat{a}^1 \sim \hat{a}^2$ to say $\hat{a}^1$ and $\hat{a}^2$ are thus related.
One can easily show that all solutions of the magnetostatic equation
\[ \tilde{C} M_\nu C \tilde{a} = \tilde{j}_{\text{src}}. \] (2.76)
are equivalent according to Definition 2.1 because for two solutions \( \tilde{a}^1 \) and \( \tilde{a}^2 \) holds \( C^T M_\nu C (\tilde{a}^1 - \tilde{a}^2) = 0 \) which directly yields \( \tilde{a}^1 \sim \tilde{a}^2 \), since \( \ker(C^T M_\nu C) = \ker(C) \).

In the case of the magnetooquasistatic curl-curl-equation not only the singularity of the curl-curl matrix \( \tilde{C} M_\nu C \) but also non-trivial nullspace of the conductivity matrix \( M_\sigma \) are the reasons for the singularity of the matrix pencil
\[ [M_\sigma, \tilde{C} M_\nu C] = \tilde{C} M_\nu C + \alpha M_\sigma. \] (2.77)

Therefore the equivalence of two solutions of the MQS curl-curl equation is not clear anymore, since non-equivalent solutions could be introduced by the nullspace of \( M_\sigma \). This leads to a numerical rather than physical problem, since we are only interested in the solutions of the equivalence class that correspond to the unique magnetic field strength. The solving of this system still makes sense and all we need is to pick out one specific solution. This leads to the following gauging approach.

**Gauging Techniques**

A gauge is an additional constraint to a non-uniquely solvable system and it removes redundant degrees of freedom in its variables to obtain a uniquely solvable system. The choice of gauge corresponds to the selection of a representative in classes of equivalent representations in the sense of Definition 2.1 [39]. The decision is by no means trivial, since assumptions have to be made and thus one gauge is not applicable to every configuration.

The particular gauge for equation (2.69) is only needed in the non-conducting regions, since there is an implicit constraint for the time-derivative of the MVP in the conducting regions
\[ \tilde{S} M_\sigma \frac{d}{dt} \tilde{a} = 0. \] (2.78)

Equation (2.78) is derived by applying the discrete divergence operator \( \tilde{S} \) on both sides of equation (2.69) with the prerequisite of an solenoidal source current density \( \tilde{j}_{\text{src}} \).

This leads directly to the most obvious idea of defining a very small artificial conductivity \( \sigma_{\text{art}} \ll 1 \) even in the non-conducting regions. The so called \( \kappa - \text{regularization} \) can easily be applied by adding a positive (semi-definite) matrix

\[ M_\sigma \rightarrow M_\sigma + \delta \kappa \]
2.3. NUMERICAL ANALYSIS

Figure 2.7: \(\kappa\)-regularization. MATLAB’s spy-plots of the matrices from the transformer example (Section 6.1) with a conducting iron core discretized using a coarse grid. Please note that the gauged matrix (b) is only regular, if the dispensable equations are disregarded (“shrinking”)

\[
M_{\sigma,art} \text{ to the discrete conductivity matrix } M_{\sigma}, \text{ which yields}
\]

\[
\tilde{C} M_\nu C \tilde{a} + (M_\sigma + M_{\sigma,art}) \frac{d}{dt} \tilde{a} = \tilde{j}_{src},
\]

(2.79)

with a regular matrix pencil \([M_\sigma + M_{\sigma,art}, \tilde{C} M_\nu C]\). The field solutions \(\tilde{b}\), which are computed with \(\tilde{b} = C\tilde{a}\) of the modified system converge in the limit \(\sigma_{art} \rightarrow 0\) to the original formulation. The regularity of \(M_\sigma + M_{\sigma,art}\) has an obvious impact on the structure of the system, since the DAE turns into a (stiff) ODE, see Fig. 2.7 for the consequences of this particular regularization. One can argue whether this is an advantage or a disadvantage, but the increased effective condition number of the new matrix pencil is a serious downside as explained in Clemens [15].

Since we do not want the DAE to become an ODE, one finds a similar numerical gauge, when adding the artificial conductivity matrix \(M_{\sigma,art}\) to the curl-curl matrix, i.e.,

\[
\left(\tilde{C} M_\nu C + M_{\sigma,art}\right) \tilde{a} + M_\sigma \frac{d}{dt} \tilde{a} = \tilde{j}_{src}.
\]

(2.80)

The matrix pencil is regular again and the equation (2.80) remains a DAE. The disadvantage of this particular gauge is the fact that it has no physical meaning at all. Furthermore we are still limited to solvers which are robust with respect to \(\sigma_{art} \rightarrow 0\) for an efficient gauging. This approach is used in the algebraic multigrid method developed by Reitzinger and Schöberl [48].
Another approach without the drawback mentioned before, but still utilizing an artificial conductance, is presented in Clemens and Weiland \[17\]. The non-uniqueness is corrected with a locally added \textit{grad-div} term

\[ M_1 G M_2 \tilde{S} M_1, \quad (2.81) \]

with an extended conductivity matrix \( M_1 \) and a matrix \( M_2 \) for scaling purposes. The matrix \( M_1 \) equals the discrete conductance matrix \( M_\sigma \), except for the zero entry on the diagonal in non-conducting regions, which are replaced by ones, in other words

\[ (M_1)_{ij} := \begin{cases} (M_\sigma)_{ij} & \text{if } i = j \text{ and } (M_\sigma)_{ij} \neq 0, \\ 1 & \text{if } i = j \text{ and } (M_\sigma)_{ij} = 0, \\ 0 & \text{else,} \end{cases} \quad (2.82) \]

for \( i, j = 1, \ldots, N_{\text{fit}} \). Then the DAE system reads with definition (2.81) of the grad-div term and the topological property (2.57)

\[ \left( \tilde{C} M_\sigma C - M_1 G M_2 \tilde{S} M_1 \right) \ddot{a} + M_\sigma \frac{d}{dt} \dot{a} = \tilde{j}_{\text{src}}. \quad (2.83) \]

This approach corresponds to the well-known \textit{Coulomb gauge} \( \nabla \cdot \vec{A} = 0 \), which guarantees a divergence-free vector potential. The discretized constraint reads for the non-conducting regions in terms of the extended conductivity matrix \( M_1 \)

\[ \tilde{S} M_1 \ddot{a} = 0. \quad (2.84) \]

The Coulomb gauge is very handy in theoretical analysis, but there is no clear numerical advantage in the discretized case \[17\]. The correspondence of the two cases can easily be seen when the constraint equation (2.84) is explicitly forced within a mixed system in \( (\vec{A}, \varphi)\)-formulation \[15\], i.e.,

\[ \begin{pmatrix} \tilde{C} M_\sigma C - M_1 \tilde{S} \\ -\tilde{S} M_1 \end{pmatrix} \ddot{a} + \begin{pmatrix} M_\sigma & 0 \\ 0 & 0 \end{pmatrix} \frac{d}{dt} \dot{a} = \begin{pmatrix} \tilde{j}_{\text{src}} \\ 0 \end{pmatrix}, \quad (2.85) \]

with \( M_2^{-1} = 0 \). Other choices of this matrix lead to different gaugings and this general approach coincides with the formulation of equation (2.70), see Clemens \[15\], since the elimination of \( \phi \) yields (2.83) and thus the regular matrix pencil reads in both cases

\[ \left[ M_\sigma, \tilde{C} M_\sigma C - M_1 G M_2 \tilde{S} M_1 \right]. \quad (2.86) \]

A different gauge of the MVP in the non-conducting regions, the so-called \textit{tree-cotree} gauging technique, comes from a graph theoretical point of view. The idea originates in circuit analysis and was adapted by Albanese and Rubinacci \[1\]. It incorporates the fact, that for a consistent representation of the MVP, one can reduce the degrees of freedom in the computational grid. It is sufficient
to relate the one degree of freedom of the magnetic field strength $\tilde{b}$ through a facet to only one degree of freedom for the path integral along the edges of this facet. This yields a decomposition of the grid in a spanning-tree and its cotree, see Fig. 2.8 for an example. The resulting gauging constraints correspond to a weakly imposed Coulomb gauging \[13\]. Thus similar systems to (2.85) arise, which have worse condition numbers than the original formulation, because the decomposition increases the spread between the largest and smallest eigenvalue, see Igarashi \[37\].

In practice the way out of the dilemma is simple, since the Krylov subspace methods converge even for singular systems, see Section 5.3.1. Furthermore if the right hand side current vector $\tilde{j}_{\text{src}}$ is consistent, those methods will not introduce irrotational parts to the solution. This property is often referred to as weak gauging.

What remains is an analytical impasse: on one hand we have to introduce the concept of gauging because the analysis of the problem will require an uniquely solvable system, but on the other hand we are interested in the properties of the original formulation and so the choice of a particular gauge would result in a loss of generality. The overview was given to illustrate the possibilities, but we shall refer to the gauge abstractly by defining an arbitrarily gauged representative of the original equation.

**Representative of the MQS Curl-Curl Equation**

All strong gauging techniques produce a regular matrix pencil by construction, but the algebraic index equals $\nu = 1$ only, as along as the nilpotency of $M_\nu$ remains unchanged. Therefore we consider only gauging techniques that exclusively modify the curl-curl matrix $\tilde{C}M_\nu C$ and we denote the (regularized) matrix
with $K_\nu$. This obviously excludes the $\kappa$-regularization.

We showed in the section on definiteness the symmetry and positive semi-definite-
ness of both the discrete matrix of conductivities $M_\sigma$ and the ungauged curl-curl
matrix $\tilde{C}M_\sigma C$. Let us consider only gauging techniques that conserve those
properties. Therefore the matrix pencil $[M_\sigma, K_\nu]$ is on the one hand regular
due to the regularization and on the other hand it is symmetric positive semi-
definite because it is the sum of two symmetric positive semi-definite matrices,
see Proposition A.1 in the Appendix. Hence we conclude its overall symmetry
and positive definiteness.

The reason for the introduction of the gauging techniques was to eliminate the
numerical problems due to the arbitrariness. We shall show, that this is finally
achieved.

If $\tilde{a}_i(t), i \in I$ denotes all solutions of the non-unique MQS curl-curl equation
\begin{equation}
M_\sigma \cdot \frac{d}{dt} \tilde{a}(t) + \tilde{C}M_\nu \cdot \tilde{a}(t) = \tilde{j}_{src}
\end{equation}
which all fulfill $\tilde{a}_i(t) \sim \tilde{a}_j(t)$ for $i, j \in I$ and $t \in [0, \tau]$ and if $\tilde{a}^R \sim \tilde{a}^i$ denotes the
unique solution of the gauged system
\begin{equation}
M_\sigma \cdot \frac{d}{dt} \tilde{a} + K_\nu \cdot \tilde{a}(t) = \tilde{j}_{src} ,
\end{equation}
then we may assume that there is for every $t \in [0, T]$ an $i(t) \in I$ such that
$\tilde{a}^R(t) = \tilde{a}^{i(t)}(t)$ holds. Hence we may conclude without loss of generality that
$\tilde{a}^R = \tilde{a}^{i_0}$ and thus chose the gauged system as the object of our further analysis.

Let us finalize this chapter with a formulation of this assumptions to which we
will refer in the following

**G1: “Regular Matrix Pencil”** The matrix pencil $[M_\sigma, K_\nu]$ of the MQS curl-
curl equation is regular.

### 2.4 Summary

In this chapter we introduced Maxwell’s equations and their discretization by the
FIT. The discrete field problem has been formulated in terms of the magnetic
vector potential and its non-uniqueness has been resolved by assuming the reg-
ularity of the matrix pencil. This allowed us to prove that the problem has an
DAE index of 1.

We now move on to the application of the electromagnetic theory to electric
circuit analysis in the next chapter, but we shall come back to the field problem
and its discretization in the chapter after next, when we establish a coupling
between field and circuit.
Chapter 3

MNA Circuit Simulation

3.1 Network Approach

The field theory of the previous chapter, which is mainly based on Maxwell’s equations (ME), is applicable to electrical circuits as well, although the large complexity of integrated electrical circuits makes simplifications unavoidable. Therefore an independent theory was deduced from ME that is tailored for circuit simulations as explained in [14]. The spatial dimensions of the elements are disregarded and one switches over from a 3-D to a 0-D description. The corresponding circuit laws for this simplified case shall be derived in the following.

A network of \( n_e \) nodes is considered, that is interconnected by \( n_b \) branches, each represents an electric (two-terminal) element. The circuit’s topology is retained in an incidence matrix \( A \)

\[
A_{kb} := \begin{cases} 
1 & \text{if branch } b \text{ leaves node } k \\
-1 & \text{if branch } b \text{ enters node } k \\
0 & \text{else}
\end{cases} \tag{3.1}
\]

with \( k = 1, \ldots, n_e \) and \( b = 1, \ldots, n_b \). It can be split into element related incidence matrices \( A = (A_C, A_L, A_R, A_V, A_I) \) for charge and flux storing elements, resistors, voltage and current sources for corresponding numbered branches. This decomposes the sum of branches into \( n_b = n_C + n_L + n_R + n_V + n_I \).

The element’s time behaviour is modelled by characteristic equations, which are basically deducible from Maxwell Equations. They relate the following integral quantities to each other

- The branch currents are defined in coherence with (ME) as integrals of the current density \( \vec{J} \) over the corresponding cross sections \( A_b \) of the branches

\[
i_b(t) := \int_{A_b} \vec{J} \, d\vec{A} \quad b = 1, \ldots, n_b . \tag{3.2}
\]
For the purpose of a coherent notation the branch currents are denoted here by small letters $i(t) = (i_1(t), \ldots, i_{n_b}(t))^T$.

- The branch voltages are the line integrals of the electric field $\vec{E}$ over a particular branch length $\ell_b$

$$u_b := \int_{\ell_b} \vec{E} \, d\ell \quad b = 1, \ldots, n_b.$$  \hfill (3.3)

Let $u(t) = (u_1(t), \ldots, u_{n_b}(t))^T$ denote the branch voltages.

- The node potentials are denoted as $e(t) = (e_1(t), \ldots, e_{n_e}(t))^T$. They correspond to the electric scalar potential $\phi$ in Section 2.1.4, but are now defined with respect to a special ground node $e_0$, which is fixed at zero. The matrix $A$ is often referred to as reduced incidence matrix, since this ground node is ignored.

The interaction of the elements is expressed in Kirchhoff’s circuit laws, which are easily derived from Maxwell’s equations in the static case, for example in [56].

**Kirchhoff’s Current Law**

Let us consider Ampère’s law (2.16b) and apply the divergence operator on both sides, which yields

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \left( \frac{\partial \vec{D}}{\partial t} + \vec{J}_{\text{cond}} \right) = 0.$$  \hfill (3.4)

The solenoidality of the total current density $\vec{J} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_{\text{cond}}$ follows naturally from the vector-analytical property (2.58); hence a closed surface integral

$$\oint_A \left( \frac{\partial \vec{D}}{\partial t} + \vec{J}_{\text{cond}} \right) \cdot d\vec{A} = 0$$  \hfill (3.5)

vanishes as well. The decomposition of this integral into the displacement and conduction current yields the continuous form of Kirchhoff’s current law (KCL)

$$\oint_A \left( \frac{\partial \vec{D}}{\partial t} + \vec{J}_{\text{cond}} \right) \cdot d\vec{A} = \int_{A_1} \frac{\partial \vec{D}}{\partial t} \cdot d\vec{A} + \int_{A_2} \vec{J}_{\text{cond}} \cdot d\vec{A}.$$  \hfill (3.6)
3.1. NETWORK APPROACH

![Kirchhoff’s Current Law](image)

Figure 3.1: Kirchhoff’s Current Law. A current entering a node $K$ is equal to the current leaving that node [56].

The displacement currents are disregarded, since they are only required if one wants to apply KCL within a capacitor as shown in Fig. 3.1(a). In circuit simulation, however, the capacitor is typically treated as a unit and one would not choose the surface integral to pass through it, see Fig. 3.1(b).

Therefore equation 3.5 states simply the conservation of the conduction currents and it reads for currents $i_b$ that flow through discrete branches $B_k \subseteq \{1, \ldots, n_b\}$, which enter the node $k$

$$\sum_{b \in B_k} i_b(t) = 0. \quad (3.7)$$

Applying equation (3.7) on every node in the circuit yields in matrix notation

$$A \cdot i(t) = 0, \quad (3.8)$$

with the incidence matrix $A \in \{-1, 0, 1\}^{n_u \times n_b}$ from above. It picks out the involved branches and the equations sets their particular sum to zero.

**Kirchhoff’s Voltage Law**

Analogously to the previous deduction a second one is derived from Faraday’s law, which reads in the electrostatic case

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0. \quad (3.9)$$

Its right hand side equals zero because changing magnetic fields are disregarded, although present in any element of an electric network. Disregarding this is the
reason for the field-circuit coupling in Section 4.1. However we apply Stoke’s Theorem to equation (3.9) and obtain the closed line integral

\[
\oint_{\ell} \vec{E} \cdot d\vec{\ell} = 0,
\]

(3.10)

where \( \ell \) denotes a loop of branches. This is Kirchhoff’s voltage law (KVL) in integral notation. It states that the algebraic sum of all branch voltages in a loop of the circuit must equal zero, since the voltage is defined as the line integral over the electric field strength. Finally the law reads expressed in terms of the incidence matrix \( A \) and node potentials \( e \)

\[
A^T \cdot e(t) = u(t).
\]

(3.11)

### 3.1.1 General Circuit Elements

The branches correspond to devices and their transient behaviour needs to be specified by characteristic equations. One speaks of general circuit elements that are composed of simple circuit elements. The latter are models of lumped two-terminal devices whose equations are in the simplest case linear relations of the introduced quantities above [14].

**(a) Charge storing elements.** The simple representative is the linear capacitor. It stores energy in an electrostatic field and is described in by a linear law. We formulate it in the general matrix notation, since the circuit may contain several capacitors

\[
q_C(A^T_C e, t) = CA^T_C e,
\]

(3.12)

where \( q_C \) is the electric charge and \( C \) is the diagonal matrix of capacitances. Each capacitor is represented by a circuit symbol of two parallel lines, see Fig. 3.2(a).
3.2 MODIFIED NODAL ANALYSIS

(b) **Flux storing elements.** The simple network element of this class is the inductor. It stores its energy in an electromagnetic field. The linear law reads for multiple inductors in matrix notation

\[ \Phi_L(jL, t) = L_{\mathbf{JL}}, \]

where \( \Phi_L \) is the magnetic flux, \( L \) is the diagonal matrix of inductances. The circuit symbol is typically a wiggly line, see Fig. 3.2(b).

(c) **Resistor.** These devices limit (“resist”) the flow of electrical current by generating voltage drops. They follow Ohm’s law in the macroscopic view, see equation (2.15)

\[ r(A_T R, t) = G_A T R, \]

where \( G = R^{-1} \) is the diagonal matrix of conductivities, while \( R \) is the analogue for the resistance. The circuit symbol is a rectangle, see Fig. 3.2(c).

(d) **Voltage source.** This element produces an electromotive force between its terminals, Fig. 3.3(a). If this force is independent of any other variable within the circuit, it is called an independent voltage source, in the other case a dependent or controlled voltage source. We consider only independent voltage sources \( v(t) \).

(e) **Current source.** This element emits electric current. Analogously to the previous definitions a current source is called dependent or controlled, if the current is affected by other circuit variables, Fig. 3.3(c) shows its symbol. If not, it is called independent, see Fig. 3.3(b). Independent current sources are denoted by \( i \) and controlled current sources by \( \lambda \). We shall concentrate on voltage-controlled current sources with parallel capacitive branches (“C-VCCS”).

3.2 Modified Nodal Analysis

We now have all the information that we need to describe the circuit: The topology is represented through incidence matrices, the physical behaviour of the elements is modelled by characteristic equations and finally Kichhoff’s circuits laws combine these relations to build a network model.
3.2.1 Conventional System

In *modified nodal analysis* (MNA) we first apply KCL to each node except ground and replace all branch currents of current defining elements by their characteristic equation as well as all branch voltages by node voltages with the help of KVL. The currents through voltage sources and inductors are added to the system as additional unknowns [32].

We finally obtain the *conventional* MNA system

\[
A_C \frac{d}{dt} q_C(A_C^T e, t) + A_R r(A_R^T e, t) + A_{LJL} + A_{VJV} + A_I(t) + A_{\lambda}(A_T e, t) = 0, \tag{3.15a}
\]

\[
\frac{d}{dt} \phi_L(J_L, t) - A_L^T e = 0, \tag{3.15b}
\]

\[
A_V^T e - v(t) = 0, \tag{3.15c}
\]

with unknown node potentials \(e\), branch currents through voltage sources and inductors \(j_V\) and \(j_L\), voltage-dependent charges of conductors and fluxes of inductors \(q_C\) and \(\phi_L\), voltage-dependent resistors \(r\), functions of time-dependent current and voltage sources \(i\) and \(v\), respectively and controlled current sources \(\lambda\).

The MNA System (3.15) can be rewritten in *charge/flux oriented* form that has several advantages, for example it assures charge conservation. A brief discussion of both approaches can be found in Feldmann and Günther [29].

3.3 Numerical Analysis

Since the behaviour of any DAE System depends heavily on its index, the concept was already used to analyze properties of the field equation. The definitions

<table>
<thead>
<tr>
<th>Unknown quantities</th>
<th>Source functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e)</td>
<td>vector of node potentials</td>
</tr>
<tr>
<td>(i_L)</td>
<td>currents through inductor</td>
</tr>
<tr>
<td>(i_V)</td>
<td>currents through voltage sources</td>
</tr>
</tbody>
</table>

Table 3.1: Electric Quantities. Overview of symbols, descriptions and units in MNA.
are given in Section B.2.2 and are easily applied to the case of a linear MNA system, since the arising system is obviously a linear implicit differential-algebraic system of the introduced type (B.7). The structure arises not only for circuits containing the linear elements defined through the characteristic equations from above, but also for originally non-linear systems, when they are linearized. This is a common approach to investigating the behaviour of non-linear circuits for small derivations, see [32]. Nevertheless we will give in the last section of this chapter a short overview of the numerical characteristics for the non-linear case but we focus in the following on the linear one.

### 3.3.1 Linear Circuits

Let the MNA system be linear, then the network (3.15) can be written as

\[
\begin{pmatrix}
ACCA^T & 0 & 0 \\
0 & L & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
d \frac{d}{dt} e \\
d \frac{d}{dt} j_L \\
d \frac{d}{dt} j_V
\end{pmatrix}
\begin{pmatrix}
e \\
j_L \\
j_V
\end{pmatrix}
+ \begin{pmatrix}
ARGA^T & AL & AV \\
0 & 0 & 0 \\
-AL^T & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e \\
j_L \\
j_V
\end{pmatrix}
= \begin{pmatrix}
-(A\dot{t} + A_\lambda \lambda) \\
0 \\
-v
\end{pmatrix},
\]

(3.16)

and it reads for short with one augmented time-dependent state variable \(x_{\text{mna}} := (e, j_L, j_V)^T\)

\[
M_{\text{mna}} \dot{x}_{\text{mna}} + K_{\text{mna}} x_{\text{mna}} = k_{\text{mna}}(t, \lambda),
\]

(3.18)

where \(M_{\text{mna}}\) and \(K_{\text{mna}}\) denote the mass and stiffness matrices from above and \(k_{\text{mna}}(t, \lambda)\) the right-hand side function.
Figure 3.4: Topological Condition $H1$. Voltage-controlled current sources with parallel capacitive branch.

Index and Uniqueness

For theoretical circuit analysis the linear implicit system (3.17) is not convenient and we prefer a separation of the algebraic and differential parts. This separation is derived constructively in terms of the MNA and not by Kronecker’s Theorem B.2. As a consequence the notations are not completely compatible. Following Estévez Schwarz and Tischendorf [28], [57] we shall accomplished our decomposition in the following by the help of a projector $Q_C$ onto the kernel of $A_C^T$, i.e.

$$Q_C \ker A_C^T = \ker A_C^T \quad \text{and} \quad A_C^T Q_C = 0, \quad (3.19)$$

and its complementary projection $P_C = I - Q_C$.

If the following topological conditions hold

**H1 ("Only C-VCCS"):** The circuit contains only voltage-controlled current sources, which are inserted into capacitive branches. This reads expressed in terms of the projector $Q_L^C A_L \lambda = 0$,

**T1 ("No VC-loops"):** The circuit contains no loops of only capacitors and voltage sources, i.e., $\ker(A_C, A_R, A_V)^T = \{0\}$,

**T2 ("No LI-cutsets"):** The circuit contains no cutsets of inductors and/or current sources, i.e., $\ker Q_L^C A_V = \{0\}$.

Index-1 systems are guaranteed in the MNA and the application of the projectors to (3.17) yields the semi-explicit system (3.20); this is manifested in the following theorem, see Tischendorf [57].

**Theorem 3.1.** If the linear network respects conditions $H1$, $T1$ and $T2$ except for capacitive-only loops, then the MNA leads to a DAE index of maximally 1.

As stated before the unknowns are divided into the differential component $y_{mna} = (P_C e, j_L)^T$ and algebraic component $z_{mna} = (Q_C e, j_V)^T$ by the help of the projectors $Q_C$ and $P_C$. The linear network equations (3.17) read in terms of those
unknowns

\[
\begin{pmatrix}
H_C & 0 \\
0 & L
\end{pmatrix}
\dot{y}_{mna} +
\begin{pmatrix}
A_R G A_T^R & A_L \\
-A_T^L & 0
\end{pmatrix}
y_{mna} +
\begin{pmatrix}
A_R G A_T^R & A_V \\
-A_T^L & 0
\end{pmatrix}z_{mna} +
\begin{pmatrix}
A_I t + A_L \lambda \\
0
\end{pmatrix} = 0 \quad (3.20a)
\]

\[
\begin{pmatrix}
Q_C^T A_R G A_T^R Q_C & Q_V^T A_V \\
A_V^T P_C & 0
\end{pmatrix}z_{mna} +
\begin{pmatrix}
Q_C^T A_R G A_T^R P_C & Q_C^T A_L \\
A_V^T P_C & 0
\end{pmatrix}y_{mna} +
\begin{pmatrix}
Q_C^T A_I t \\
-\nu
\end{pmatrix} = 0 \quad (3.20b)
\]

with a regular matrix \( H_C := A_C C A_C^T + Q_C^T Q_C \), which is introduced for notation purposes only.

Now one proves Theorem 3.1 by showing that under the topological conditions \( H_1, T_1 \) and \( T_2 \) the algebraic part (b) of system (3.20) is solvable for \( z_1 \), i.e.,

\[
P_{mna} := \begin{pmatrix}
Q_C^T A_R G A_T^R Q_C & Q_V^T A_V \\
A_V^T P_C & 0
\end{pmatrix}
\]

is regular. Hence \( z_{mna} \) is a function of the differential part \( y_{mna} \) and of the source functions \( \iota \) and \( \nu \) and thus differential index 1 holds, since only one differentiation leads to an ODE, see Section 3.2.2 for details on the index concept.

Since \( H_C \) and \( L \) are regular, equation (3.20a) is an ODE in \( y_{mna} \) for a given \( z_{mna} \). Let the initial value be denoted by

\[
y_{mna}(0) = y_{mna,0} = \begin{pmatrix}
P_C e_0 \\
J_L, 0
\end{pmatrix}.
\]

Then the algebraic equation (3.20b) is fixed and the overall initial value reads

\[
x_{mna,0} = \begin{pmatrix}
P_C e_0 + Q_C e_0 \\
J_L, 0 \\
J_V, 0
\end{pmatrix}.
\]

Finally the topological conditions guarantee a unique continuous solution on the considered time interval \([0, \tau]\), because the ODE part is uniquely solvable if the right-hand sides are Lipschitz continuous on the interval, according to Theorem B.1 of Picard-Lindelöf.

Problems of higher index than 2 can be avoided if the conditions \( T_1, T_2 \) and \( H_1 \) are considered in the circuit design. This is relevant since the cases which
should be prevented are by no means pathological. Index-2 systems arise even for simple circuits of the linear class. The circuits of Fig. 3.5 are generic examples of index-2 problems, that are forbidden by the topological conditions. The first example in Fig. 3.5(a) contains a loop of a voltage source and a capacitor, while the second example in Fig. 3.5(b) includes a cutset of an inductor and a current source.

Non-Linearity

As mentioned before the numerical properties of non-linear MNA systems are often studied by their linearized system and thus the results regarding uniqueness and DAE index of the previous section are simply transferred. On the other hand the same conditions can be shown to apply to non-linear systems as well, if the generalized matrices

\[
L(i) = \frac{\partial \Phi(w)}{\partial w}, \quad C(u) = \frac{\partial q(w)}{\partial w}, \quad G(u) = \frac{\partial r(w)}{\partial w}
\]

are positive definite for of all inductance, capacitance and resistances, respectively. If finally the topological conditions H1, T1 and T2 hold, then the MNA still leads to an index-1 DAE in conventional and charge/flux oriented form [57].

3.4 Summary

We have discussed how circuits that contain the common devices from above are mathematically expressed using the MNA. The numerical properties have been derived and we proved that under given topological conditions the circuit problem is a DAE of index 1.

The material relations of the elements were given for linear elements and one could see that they do model only specific effect and do not cover all phenomena satisfyingly. Therefore we shall derive in the next chapter more sophisticated models that refine the inductor element.
Chapter 4

FIT Models for the Circuit Simulation

4.1 Conductor Models

The discretization described in Section 2.2 can be applied to a wide range of devices. In our focus are device parts that are typically embedded in electrical circuits as given in Chapter 3. Theoretically, magnetic effects are present in any element of an electric network. In many cases they can be neglected or they can be modeled by a few inductances inserted in the circuit. However, for complicated geometries, the construction of circuit models for parts with mutual magnetic interaction or parts with particular eddy-current effects, is tedious. It then makes sense to introduce a field model and to identify some conductors appearing in the field model as branches in the circuit. In the following two substituting models are derived: the solid conductor and stranded conductor, see Fig. 4.1. They will be inserted in our circuit simulations later on, utilizing the MQS curl-curl equation and additional coupling equations as given in [22, 9].

Besides the circuit model, one or several field models will be considered. In this treatise, the field models are discretized by the FIT. In each field model, an arbitrary number of solid conductors and stranded conductors is present. Each of these solid and stranded conductors may be connected to the circuit model as shown below. For the derivation of the coupling, a single circuit that contains only a current or voltage source is connected with its two ports to a single field model in which one solid conductor or one stranded conductor is present, respectively. This particular configuration is exemplary and will be generalized later on.

Let $\Omega_{\text{FIT}}$ denote the whole computational domain, $\Gamma_{\text{sol}}$ and $\Gamma_{\text{str}}$ cross sections through the solid and stranded conductor, respectively. The quantities $\ell_{\text{sol}}$ and $\ell_{\text{str}}$ represent the length of a particular conductor. The same distinction is made
4.1.1 Solid Conductor Model

In a massive conductor, the current density is not equally distributed for high frequencies. There is a tendency for the current density in the core of the conductor to be smaller than near the surface. This phenomenon is called skin effect, see Fig. 4.2. It causes the resistance of the conductor to increase with the frequency of the current. For direct current (dc) this effect does not occur and the current is equally distributed. A similar phenomenon appears in a solid conductor when localized in the neighbourhood of other current carrying conductors. Also then, eddy currents and eddy-current losses appear in the solid conductor. This effect is called proximity effect. The reason for both phenomena is the existence of a magnetic field which affects the applied field [31, Section 7.3.11]. This effect is to be simulated in the following: the solid conductor will serve as the device in a electrical circuit, where skin- and proximity effects are considered.

The current density \( \mathbf{j}_{\text{sol}} \) in a solid conductor is decomposed in the source current density \( \mathbf{j}_{\text{src}} \) and the eddy current density \( \mathbf{j}_{\text{eddy}} \). The decomposition into the potentials is not unique, which is a consequence of their definition, and only the sum is a measurable quantity. The total current is calculated with the help of
4.1. CONDUCTOR MODELS

Figure 4.2: Skin Effect. There is a tendency so that the current density in the core of a solid conductor is smaller than near to the surface; this figures is similar to the one in [31].

Ohm’s law, i.e.,

\[ \vec{J}_{\text{sol}} = \vec{J}_{\text{src}} + \vec{J}_{\text{eddy}} = \sigma \vec{E} = \sigma \left( -\nabla \varphi - \frac{\partial}{\partial t} \vec{A} \right), \]

which transfers as given by De Gersem and Weiland [21] into the FIT notation utilizing an isotropic conductivity matrix \( M_{\sigma,\text{iso}} \)

\[ \vec{j}_{\text{sol}} = \vec{j}_{\text{src}} + \vec{j}_{\text{eddy}} = M_{\sigma,\text{iso}} \vec{e} = M_{\sigma,\text{iso}} \left( -G\varphi - \frac{d}{dt} \vec{a} \right). \] (4.1)

Different models for a solid conductor can be derived depending on the chosen decomposition above. In the following, two approaches are presented whose systems have the same structure but differ mainly in the sparsity of the coupling matrices.

**Coupling at Reference Cross-Section**

The choice of the electric scalar potential \( \varphi \) is the key to the different approaches. A first choice establishes rather a two-dimensional coupling than a full three-dimensional coupling, since one spatial direction is disregarded and only one layer of the grid is used as a representation of the solid conductor for the coupling. Therefore a cross-section \( \Gamma_{\text{sol}} \) is chosen such that it coincidences with an
agglomeration of facets of the dual grid, see Fig. 4.3. The primary cells, which are crossed by this section are called the reference layer.

Since the potentials do not have to be continuous - this is just required for the derived quantities - we can choose $\varphi$ piecewise constant, such that $\nabla \varphi$ represents a jump at the cross section $\Gamma_{\text{sol}}$ but still fulfills

$$U_{\text{sol}} = - \int_{\gamma_{\text{sol}}} \nabla \varphi \cdot d\vec{s}, \quad (4.2)$$

where $U_{\text{sol}}$ describes the voltage drop and $\gamma_{\text{sol}}$ is an arbitrary path through the conductor from one circuit-connected side ("port") to the other s.t. $\Gamma_{\text{sol}}$ and $\gamma_{\text{sol}}$ are intersecting in one point $[21]$.

The voltage drop $u_{\text{sol}}$ can be described in FIT as the potential difference of the top and bottom of the reference plane. This is represented by an artificial electric field quantity

$$\vec{e}_{\text{app}} = \hat{Q}_{\text{sol}} u_{\text{sol}}, \quad (4.3)$$

with an incidence vector $\hat{Q}_{\text{sol}} \in \{-1, 0, 1\}^N$ which applies the voltages to the corresponding edges of the primary grid. This establishes a coupling between the zero-dimensional voltage drop and a field distribution with a two-dimensional support.
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The total current is the integral over the cross section $\Gamma_{\text{sol}}$ in the continuous case

$$I_{\text{sol}} = \int_{\Gamma_{\text{sol}}} \vec{J}_{\text{sol}} \cdot d\vec{S}$$

(4.4)

$$= \int_{\Gamma_{\text{sol}}} -\sigma \nabla \varphi \cdot d\vec{S} - \int_{\Gamma_{\text{sol}}} \sigma \frac{\partial}{\partial t} \vec{A} \cdot d\vec{S},$$

(4.5)

while in the FIT model summarizing the contributions of the source densities at the dual facets associated with the primary edges leads to the corresponding result

$$i_{\text{sol}} = \hat{Q}_{\text{sol}}^T \vec{j}_{\text{sol}}$$

$$= \hat{Q}_{\text{sol}}^T (\vec{j}_{\text{src}} + \vec{j}_{\text{eddy}})$$

$$= \hat{Q}_{\text{sol}}^T M_{\sigma,\text{iso}} \hat{Q}_{\text{sol}} u_{\text{sol}} - \hat{Q}_{\text{sol}}^T M_{\sigma,\text{iso}} \frac{d}{dt} \hat{a},$$

(4.6)

where the value $\hat{G}_{\text{sol}}$ can be interpreted as the dc conductance of the reference layer, the resistance is obtained by inverting $R_{\text{sol}} = \frac{1}{\hat{G}_{\text{sol}}}$. The variable $\hat{a}$ denotes the MVP, which is assigned the additional diacritic to strengthen its relation to this particular choice of $\varphi$. We shall give another choice later on, still following [8].

Consider a coupled model with a single solid conductor in the field model part. When the voltage drop along the solid conductor is known a-priori, the field model described by the curl-curl equation (2.69) is excited by the source current density

$$\vec{j}_{\text{src}} = M_{\sigma,\text{iso}} \hat{e}_{\text{app}} = M_{\sigma,\text{iso}} \hat{Q}_{\text{sol}} u_{\text{sol}}.$$  

(4.7)

When, however, not the voltage drop but the current is prescribed, the voltage drop is treated as an additional unknown and the circuit coupling equation (4.6) is added to the system (2.69), yielding (4.8).

$$\begin{pmatrix} K_{\nu} & -M_{\sigma,\text{iso}} \hat{Q}_{\text{sol}} \\ 0 & \hat{G}_{\text{sol}} \end{pmatrix} \begin{pmatrix} \hat{a} \\ u_{\text{sol}} \end{pmatrix} + \begin{pmatrix} M_{\sigma,\text{iso}} \\ -\hat{Q}_{\text{sol}}^T M_{\sigma,\text{iso}} \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \hat{a} \\ u_{\text{sol}} \end{pmatrix} = \begin{pmatrix} 0 \\ i_{\text{sol}} \end{pmatrix},$$

(4.8)

**Coupling at the Conductor Volume**

The previous coupling includes the non-solenoidal source current $\vec{j}_{\text{src}} = M_{\sigma,\text{iso}} \hat{e}_{\text{app}}$ and lead to a non-solenoidal eddy current $\vec{j}_{\text{eddy}} = M_{\sigma,\text{iso}} \frac{d}{dt} \hat{a}$. This decomposition of the solenoidal current into non-solenoidal parts is sometimes cumbersome [21]. On the other hand it is possible to design a field-circuit coupling where $\vec{j}$ is
decomposed in two solenoidal current fields $\vec{j}_{\text{src}}$ and $\vec{j}_{\text{eddy}}$. We simply choose the electric scalar potential such that the source current density becomes solenoidal. Hence we postulate:

$$\nabla \cdot \vec{j}_{\text{src}} = -\nabla \cdot (\sigma \nabla \varphi) = 0. \quad (4.9)$$

This will give us the source current density $\vec{j}_{\text{src}} = M_{\sigma,\text{iso}} e_{\text{app}} + M_{\sigma,\text{iso}} \vec{S}^T \phi$ derived from the solution $\phi$ of the equivalent discretized problem

$$\tilde{S} M_{\sigma,\text{iso}} \vec{S}^T \phi = -\tilde{S} M_{\sigma,\text{iso}} e_{\text{app}}, \quad (4.10)$$

utilizing the property $G = -\tilde{S}^T$ of the discrete gradient matrix. We are now able to define a new operator $Q_{\text{sol}} \in \mathbb{R}^N$ such that $\vec{j}_{\text{src}} = M_{\sigma,\text{iso}} Q_{\text{sol}} u_{\text{sol}}$ holds, with the property $\tilde{S} \vec{j}_{\text{src}} = 0$, i.e., the discrete source current is divergence free.

It couples the voltage drop to a three-dimensional field distribution allocated at all primary edges inside the conductor volume. This is no longer an incidence vector. It contains real values and leads to a relatively dense coupling.

A field model with a single solid conductor excited by a voltage source is still described by (2.69) whereas the same field model with a current excitation leads to the system of equations (4.11) where a circuit relation serves as additional equation. The result is similar to the previous system (4.8) and reads

$$
\begin{pmatrix}
K_{\nu} & -M_{\sigma,\text{iso}} Q_{\text{sol}} \\
0 & G_{\text{sol}}
\end{pmatrix}
\begin{pmatrix}
\vec{a} \\
u_{\text{sol}}
\end{pmatrix}
+
\begin{pmatrix}
M_{\sigma,\text{iso}} & 0 \\
-Q_{\text{sol}}^T M_{\sigma,\text{iso}} & 0
\end{pmatrix}
\frac{d}{dt}
\begin{pmatrix}
\vec{a} \\
u_{\text{sol}}
\end{pmatrix}
=
\begin{pmatrix}
0 \\
u_{\text{sol}}
\end{pmatrix},
\quad (4.11)
$$

with the three-dimensional coupling matrix $Q_{\text{sol}}$ and the dc conductance matrix $G_{\text{sol}} = Q_{\text{sol}}^T M_{\sigma,\text{iso}} Q_{\text{sol}}$, which might still be inverted to yield the resistance matrix $R_{\text{sol}} = G_{\text{sol}}^{-1}$. The magnetic vector potentials $\vec{a}$ and $\vec{a}$ are not identical in both approaches, but the matrices $Q_{\text{sol}}$ and $Q_{\text{sol}}$ are connected through a projection operator $N_{\vec{e}}$, so that $Q_{\text{sol}} = N_{\vec{e}} \tilde{Q}_{\text{sol}}$ holds and thus the solutions are transformable, see De Gersem and Weiland [21].

### 4.1.2 Stranded Conductor

In contrast to the previous model, the stranded conductor model is not built of a single solid material, but consists of thin individual strands wound to form a coil, as depicted in the Fig. 4.1(b). Each strand does not exhibit significant eddy currents because of its cross section, which is assumed to be substantially smaller.
than the skin depth related to the frequencies occurring in the model. We assume furthermore a homogeneous current distribution, which corresponds to a winding with a constant cross-section.

The particular restrictions to the current distribution introduced by the interturn insulation is modelled by an anisotropic conductivity matrix $M_{\sigma,\text{aniso}}$ aligned with the winding direction replacing the isotropic conductivity matrix $M_{\sigma,\text{iso}}$ associated with the solid conductor. In the general case this matrix $M_{\sigma,\text{aniso}}$ is not diagonal anymore. It is only diagonal in the special case where the winding is aligned with one of the grid coordinates. Then, zeros are introduced at the diagonal positions corresponding to the directions in the stranded conductor perpendicular to the winding direction. These zeros model the insulation between the individual strands which prevents currents migrating between the strands. In general, however, the winding direction is not aligned with a grid direction. Then, $M_{\sigma,\text{aniso}}$ is not diagonal. The non-zero blocks in $M_{\sigma,\text{aniso}}$ are not of full rank. The rank deficiency corresponds to the directions perpendicular to the winding direction.

Roughly speaking, a non-zero block in $M_{\sigma,\text{aniso}}$ corresponding to a single stranded conductor in the model, has approximately one third non-zero eigenvalues associated with eigenvectors which correspond to current distributions aligned with the winding direction and two third zero eigenvalues associated with eigenvectors which correspond to current distributions perpendicular to the winding directions, see De Gersem and Weiland [21].

The use of a stranded conductor model alleviates the need to resolve the individual strands in the model. Similarly to the solid conductor case, a coupling can be established at a reference cross section perpendicular to the strands and at all dual facets, see [20].

Sparse Coupling at Coil Cross-Section

The average current density at the reference cross-section reads

$$\vec{j}_{\text{src}} = \frac{N_t}{\Gamma_{\text{str}}} \underline{D_{Q,\text{str}}} \hat{\underline{Q}_{\text{str}}} \hat{\underline{i}}_{\text{str}},$$

(4.12)

where $D_{\hat{A}}$ is the diagonal matrix of dual-facet areas, $N_t$ is the number of strands, $\Gamma_{\text{str}}$ is the area of the reference cross-section and $i_{\text{str}}$ is the current applied. An application of Ohm’s law leads to the voltage drop

$$u_{\text{str}} = \hat{Q}_{\text{str}}^T \left( M_{\sigma,\text{iso}} \vec{j}_{\text{src}} + \frac{d}{dt} \hat{\dot{a}} \right),$$

(4.13)

$^1M^+$ denotes the pseudoinverse of $M$, see Definition A.1.
which yields with equation (4.12)

$$u_{str} = \hat{R}_{str} i_{str} + \hat{Q}^T_{str} \frac{d}{dt} \hat{a},$$  

(4.14)

where $\hat{a}$ is the MVP for the sparse coupling at coil cross-section and $\hat{R}_{str} = \hat{Q}^T_{str} M_{\sigma,aniso}^{+} \hat{Q}_{str}$ corresponds to the dc resistance and its inverse to the conductance $\hat{G}_{str} = R_{str}^{-1}$.

The result (4.14) is completed once again with the magnetoquasistatic curl-curl equation (2.69). When the current flux through the solid conductor is known the field a coupled model with a single stranded conductor is described by (2.69) with the source current density from (4.12) on the right-hand side of the curl-curl equation. In the case where the current rather than the voltage drop is prescribed, equation (4.14) is added and the voltage drop is treated as additional unknown, so the system reads

$$\begin{pmatrix} K_\nu & -\hat{Q}_{str} \\ 0 & \hat{R}_{str} \end{pmatrix} \begin{pmatrix} \hat{a} \\ i_{str} \end{pmatrix} + \begin{pmatrix} M_{\sigma,aniso} & 0 \\ \hat{Q}^T_{str} & 0 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \hat{a} \\ i_{str} \end{pmatrix} = \begin{pmatrix} 0 \\ u_{str} \end{pmatrix}. $$  

(4.15)

**Preprocessed Dense Coupling**

The anisotropic conductivity matrix $M_{\sigma,aniso}$ is not diagonal and this decreases the efficiency of any solution algorithm, that depends on the matrix pencil, since a linear system with the matrix $K_\nu + \alpha M_{\sigma,aniso}$ has to be solved. Therefore one turns to a relatively dense coupling that avoids this term [21]. In this approach the anisotropic conductivity matrix $M_{\sigma,aniso}$ is only used in the preprocessing step

$$\tilde{S} M_{\sigma,aniso} \tilde{S}^T \phi = -\tilde{S} \tilde{j}_{app},$$  

(4.16)

which yields the divergence-free source current field

$$\bar{j}_{src} = \hat{j}_{app} + M_{\sigma,aniso} \bar{S}^T \phi = Q_{str} i_{str}. $$  

(4.17)

These prearrangements allow a radical simplification of the coupled system (4.18), so that the anisotropic conductivity matrix vanished in the MQS curl-curl equation. The coupled system of equations reads

$$\begin{pmatrix} K_\nu & -\hat{Q}_{str} \\ 0 & \hat{R}_{str} \end{pmatrix} \begin{pmatrix} \hat{a} \\ i_{str} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \hat{Q}^T_{str} & 0 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \hat{a} \\ i_{str} \end{pmatrix} = \begin{pmatrix} 0 \\ u_{str} \end{pmatrix}, $$  

(4.18)

with $R_{str} = Q^T_{str} M_{\sigma,aniso} Q_{str}$ which equals the resistance of the stranded conductor and $\hat{G}_{str} = R_{str}^{-1}$ is the conductance.

Similar to the solid conductor case there are projectors which allow a mapping between $Q_{str}$ and $\hat{Q}_{str}$ and the current densities, see De Gersem and Weiland [21].
4.1. CONDUCTOR MODELS

(a) Circuit symbol. (b) Equivalent circuit

Figure 4.5: Stranded Conductor Symbol. (a) circuit symbol and (b) equivalent circuit for a single stranded conductor

**Equivalent Circuit**

The behaviour of the above discussed case of a field model that contains only a single stranded conductor made of linear materials is completely described by the series connection of a resistance \( R_{\text{str}} = Q_{\text{str}}^T M_{\sigma,\text{aniso}}^+ Q_{\text{str}} \) and an inductance \( L_{\text{str}} = Q_{\text{str}}^T K_{\nu}^+ Q_{\text{str}} \), see Fig. 4.5(b).

In other words there is no clear advantage on traditional circuit analysis so far, but the field model allows us to include several solid and stranded conductors within on system of conductors and all magnetic phenomena that emerge for example through the proximity effect are regarded in this model. We shall present this generalization in the next section.

**4.1.3 System of Conductors**

The field model can contain several solid conductors and several stranded conductors. For the particular case where the field model contains a current-driven solid conductor and a voltage-driven stranded conductor, the MQS curl-curl equation is excited by

\[
\bar{j}_{\text{src}} = M_{\sigma,\text{iso}} Q_{\text{sol}} u_{\text{sol}} + Q_{\text{str}} i_{\text{str}},
\]

where \( u_{\text{sol}} \) and \( i_{\text{str}} \) are unknowns that are described by the additional coupling equations (4.14) and (4.14). The system of equations reads

\[
\begin{pmatrix}
K_{\nu} & -M_{\sigma,\text{as}} Q_{\text{sol}} & -Q_{\text{str}} \\
0 & G_{\text{sol}} & 0 \\
0 & 0 & R_{\text{str}}
\end{pmatrix}
\begin{pmatrix}
\bar{a} \\
u_{\text{sol}} \\
i_{\text{str}}
\end{pmatrix}
\begin{pmatrix}
\bar{a} \\
u_{\text{sol}} \\
i_{\text{str}}
\end{pmatrix}

+ \begin{pmatrix}
-M_{\sigma,\text{as}} & 0 & 0 \\
-Q_{\text{sol}}^T M_{\sigma,\text{as}} & 0 & 0 \\
-Q_{\text{str}}^T & 0 & 0
\end{pmatrix}
\frac{d}{dt}
\begin{pmatrix}
\bar{a} \\
u_{\text{sol}} \\
i_{\text{str}}
\end{pmatrix}
= \begin{pmatrix}
0 \\
u_{\text{sol}} \\
i_{\text{str}}
\end{pmatrix},
\]

with the assembled conductivity matrix \( M_{\sigma,\text{as}} \in \mathbb{R}^{N_{\text{as}} \times N_{\text{fit}}} \) that includes both entries for the solid and the stranded conductor model. The entries correspond
CHAPTER 4. FIT MODELS FOR THE CIRCUIT SIMULATION

to those of the isotropic matrix $M_{\sigma,\text{iso}}$ in the solid case and equal zero for the stranded conductor and thus it is still diagonally shaped and positive (semi-) definite.

We give in addition to the mixed current- and voltage-driven formulation (4.20) an equivalent voltage-driven reformulation in system (4.21). It is more convenient for the FIT/MNA coupling since the model can be added as one voltage-controlled (multiport) device in the MNA network. The circuit symbol for this device is a combination of the solid and stranded conductor symbols and is given in Fig. 4.6(a).

\[
\begin{pmatrix}
K_v & 0 & -Q_{\text{str}} \\
0 & I & 0 \\
0 & 0 & R_{\text{str}}
\end{pmatrix}
\begin{pmatrix}
\vec{a} \\
\iota_{\text{sol}} \\
\iota_{\text{str}}
\end{pmatrix}
+ \begin{pmatrix}
M_{\sigma,\text{as}} & 0 & 0 \\
0 & 0 & 0 \\
Q_{\text{sol}}^T & M_{\sigma,\text{as}} & 0 \\
Q_{\text{str}}^T & 0 & 0
\end{pmatrix}
\frac{d}{dt}
\begin{pmatrix}
\vec{a} \\
\iota_{\text{sol}} \\
\iota_{\text{str}}
\end{pmatrix}
= \begin{pmatrix}
M_{\sigma,\text{as}}Q_{\text{sol}}u_{\text{sol}} \\
G_{\text{sol}}u_{\text{sol}} \\
u_{\text{str}}
\end{pmatrix},
\]

(4.21)

For notation proposes we abbreviate system (4.21) and rearrange the equation to have the differential term at first position, this gives

\[
M_{\text{fit}}\dot{x}_{\text{fit}} + K_{\text{fit}}x_{\text{fit}} = k_{\text{fit}}(u_{\text{fit}}),
\]

(4.22)

with the augmented coupling matrices $M_{\text{fit}}$ and $K_{\text{fit}}$ and the new state variable $x_{\text{fit}} := (\vec{a}, \iota_{\text{sol}}, \iota_{\text{str}})^T$ that includes the unknown MVP and the currents. Let us furthermore combine the voltages to one assembled quantity $u_{\text{fit}} := (u_{\text{sol}}, u_{\text{str}})^T$.

In more complicated situations where the solid and stranded conductors are coupled through an external electric network, an aditional MNA description of the network is added, as explained in the next chapter.

Multiple Conductors

The notations from above even allows naturally multiple conductors of both types. Let $N_{\text{sol}}$ denote the number of solid and $N_{\text{str}}$ the number of stranded conductors. Then the field model acts as a multiport device with $2 \cdot (N_{\text{sol}} + N_{\text{str}})$ terminals in the electrical network, Fig. 4.6(a) shows its circuit symbol and Fig. 4.6(b) is an exemplary sketch of a field system with multiple conductors.

The coupling vectors $Q_{\text{sol}}$ and $Q_{\text{str}}$ generalize to matrices of dimension $N_{\text{fit}} \times N_{\text{sol}}$ and respectively $N_{\text{fit}} \times N_{\text{str}}$

\[
Q_{\text{sol}} = (Q_{\text{sol,p}}), \quad \text{with } p = 1, \ldots, N_{\text{sol}},
\]

(4.23)

\[
Q_{\text{str}} = (Q_{\text{str,q}}), \quad \text{with } q = 1, \ldots, N_{\text{str}}.
\]

(4.24)
4.2. NUMERICAL ANALYSIS

We discuss in this section the voltage-driven formulation (4.21), the properties of the mixed system (4.20) are analogous since similar arguments apply to both cases. We prefer the voltage-controlled formulation here because it becomes advantageous later when the coupled system is studied.

4.2.1 Algebraic Properties

Most properties follow from the fact that the 3-D conductors are spatial disjoint since this is reflected in the FIT model, as long as the discretization grid is sufficiently fine. If the resolution is too coarse the particular conductor models might be smeared into each other and the spatial disjunction is not correctly resolved in the model. See Fig. 4.7 for an schematic sketch of such a situation. The smearing effects of different materials within one conductor were already
known for coarse grids but the particular smearing of conductor types has not been studied so far.

In practice the requirement of spatial disjointness in FIT is no handicap, since the grid has to fulfill a certain amount of fineness anyway to prevent short circuits. Additionally the validity of the following properties can be manually checked with matrix-vector multiplications and if necessary the discretization can be repeated with an refined grid.

The first algebraic property concerns the matrices of conductivity. The product of the assembled conductivity matrix and the anisotropic one equals zero

\[ M_{\sigma,\text{aniso}} \cdot M_{\sigma,\text{as}} = 0, \]

\[ M_{\sigma,\text{aniso}}^+ \cdot M_{\sigma,\text{as}} = 0. \]

This has also consequences for the coupling matrices. Matrix \( Q_{\text{sol}} \) couples only to conducting regions that belong to a solid conductor which yields the property

\[ M_{\sigma,\text{as}}^+ \cdot M_{\sigma,\text{as}} \cdot Q_{\text{sol},p} = Q_{\text{sol},p}, \quad \text{for } p \in \{1, \ldots, N_{\text{str}}\}. \]

The stranded conductor case is different since all entries that belong to a stranded conductor equal zero in \( M_\sigma \). As a consequence a multiplication with \( Q_{\text{str}} \) yields the property

\[ M_{\sigma,\text{as}} \cdot Q_{\text{str},q} = 0, \quad \text{for } q \in \{1, \ldots, N_{\text{str}}\}. \]
4.2. NUMERICAL ANALYSIS

Properties hold for the multiplication of coupling matrices

\[ Q^T_{\text{sol},p} \cdot Q_{\text{sol},q} = 0, \quad \text{for all } p \neq q \text{ and } p, q \in \{1, \ldots, N_{\text{sol}}\}; \]  
\[ Q^T_{\text{str},p} \cdot Q_{\text{str},q} = 0, \quad \text{for all } p \neq q \text{ and } p, q \in \{1, \ldots, N_{\text{str}}\}; \]  
\[ Q^T_{\text{sol},p} \cdot Q_{\text{str},q} = 0, \quad \text{for all } p \in \{1, \ldots, N_{\text{sol}}\} \text{ and } q \in \{1, \ldots, N_{\text{str}}\}. \]  

(4.29)  
(4.30)  
(4.31)

After given those properties, we can substantiate the needed degree of fineness: We will call a discretization grid of the FIT sufficiently fine in the following, if the resulting matrices fulfill the properties (4.25) – (4.31). Let us assume this case in the following, if the spatial discretization condition G2 is postulated.

**G2: “Sufficiently Fine Grid”** The discretization grid is sufficiently fine.

### 4.2.2 Matrix Pencil

The matrix pencil of the coupled system (4.21) reads in dependence of the parameter \( \alpha \in \mathbb{R} \)

\[
[M_{\text{fit}}, K_{\text{fit}}] = \begin{pmatrix} K_{\nu} + \alpha M_{\sigma,\text{as}} & 0 & -Q_{\text{str}} \\ \alpha Q^T_{\text{sol}} M_{\sigma,\text{as}} & I & 0 \\ \alpha Q^T_{\text{str}} & 0 & R_{\text{str}} \end{pmatrix}.
\]

(4.32)

Let the curl-curl equation (2.70) be regularized through a gauging technique as described in Section 2.3. This guarantees by construction an \( \alpha_0 \) such that

\[
\det (K_{\nu} + \alpha_0 M_{\sigma,\text{as}}) \neq 0,
\]

(4.33)

where \( K_{\nu} + \alpha_0 M_{\sigma,\text{as}} \) is symmetric and positive definite if the gauging conserves the corresponding properties of \( M_{\sigma,\text{as}} \) and \( CM_{\nu}C \). We assume furthermore without loss of generality that \( \alpha_0 > 0 \) holds because of Proposition B.1.

We will show in the following that the matrix pencil (4.32) is regular for the given parameter \( \alpha_0 > 0 \). Therefore we eliminate \( -\alpha Q^T_{\text{sol}} M_{\sigma,\text{as}} \) and \( \alpha Q^T_{\text{str}} \) by two Gaussian elimination steps, which leave the determinant unchanged but form an upper triangular matrix. Then the overall determinant equals the product of the determinants of the three block matrices on the diagonal. The regularity of both \( K_{\nu} + \alpha_0 M_{\sigma,\text{as}} \) and the identity \( I \) is clear. The only questionable determinant reads

\[
\det \left( R_{\text{str}} + \alpha_0 Q^T_{\text{str}} (K_{\nu} + \alpha_0 M_{\sigma,\text{as}})^{-1} Q_{\text{str}} \right) \neq 0.
\]

(4.34)

Both \( R_{\text{str}} \) and \( K_{\nu} + \alpha_0 M_{\sigma,\text{as}} \) are diagonal and positive definite thus the same holds true for their inverse matrices. Furthermore the matrix \( Q_{\text{str}} \) has only a trivial kernel and thus the matrix product \( Q^T_{\text{str}} (K_{\nu} + \alpha_0 M_{\sigma,\text{as}})^{-1} Q_{\text{str}} \) is symmetric.
positive definite according to Proposition A.4. Therefore the determinant in inequation (4.34) does not vanish since the sum of two symmetric positive definite matrices is symmetric positive definite and hence regular.

The second term within the determinant operator has a physical meaning for a configuration with only stranded conductors and without other conductive parts. The positive definite term \( Q_{\text{str}}^T K_{\nu}^+ Q_{\text{str}} \) equals the inductance matrix

\[
L_{\text{str}} := Q_{\text{str}}^T K_{\nu}^+ Q_{\text{str}}
\]

containing the self-inductances \( L_{\text{str},i,i} \) and the mutual inductances \( L_{\text{str},i,j} \) between the stranded conductors \( (i \neq j) \). Note that the pseudoinverse equals the natural inverse, i.e., \( K_{\nu}^+ = K_{\nu}^{-1} \) only if the gauging yields a regular matrix \( K_{\nu} \).

### 4.2.3 DAE Index

The DAE index of the original curl-curl equation (2.69) was shown by examining the nilpotency of the constant coefficient matrix \( M_{\sigma,\text{as}} \). Let us apply the same technique to system (4.21) and therefore transform the system into a more convenient formulation.

The transformation is applied utilizing the following regular matrix that corresponds to block-Gaußian eliminations

\[
R = R_1 \cdot R_2 := \left( \begin{array}{ccc}
I & 0 & 0 \\
-Q_{\text{sol}}^T & I & 0 \\
-Q_{\text{sol}}^T M_{\sigma,\text{aniso}}^+ & 0 & I
\end{array} \right) \cdot \left( \begin{array}{ccc}
I & 0 & Q_{\text{str}} G_{\text{str}} \\
0 & I & 0 \\
0 & 0 & I
\end{array} \right)
\]

\[
= \left( \begin{array}{ccc}
I & 0 & Q_{\text{str}} G_{\text{str}} \\
-Q_{\text{str}}^T & I & 0 \\
-Q_{\text{str}}^T M_{\sigma,\text{aniso}}^+ & 0 & 0
\end{array} \right),
\]

where \( R_2 \) corresponds to a elimination of \(-Q_{\text{str}}\) in the first equation of (4.21) and \( R_1 \) removes the dependence on the derivative of the MVP in the second and third equation.

After a left multiplication of system (4.21) by \( R \) and utilizing the properties of the coupling matrices the voltage-dependent system reads

\[
(M_{\sigma,\text{as}} + Q_{\text{str}} G_{\text{str}} Q_{\text{str}}^T) \frac{d}{dt} \vec{\alpha} + K_{\nu} \vec{\alpha} = \underbrace{\vec{J}_{\text{src}}}_{\text{jsrc}} + M_{\sigma,\text{as}} Q_{\text{sol}} u_{\text{sol}} + Q_{\text{str}} G_{\text{str}} u_{\text{str}},
\]

\[
u_{\text{sol}} = Q_{\text{sol}}^T K_{\nu} \vec{\alpha},
\]

\[
u_{\text{str}} = G_{\text{str}} Q_{\text{str}}^T M_{\sigma,\text{aniso}}^+ K_{\nu} \vec{\alpha},
\]
where the source current density \( \tilde{j}_{\text{src}} := M_{\sigma,\text{as}} Q_{\text{sol}} u_{\text{sol}} + Q_{\text{str}} G_{\text{str}} u_{\text{str}} \) corresponds to the direct current, that would flow if no magnetic effects are present or the frequency is zero.

System (4.37) has still a regular matrix pencil since it is a result of system (4.21) that is obtained by a left multiplication with the regular matrix \( R \). The algebraic DAE index equals 1 because the mass matrix is symmetric, which is trivial since it is the sum of to symmetric matrices

\[
M_{\sigma,\text{as}} + Q_{\text{str}} G_{\text{str}} Q_{\text{str}}^T = (M_{\sigma,\text{as}} + Q_{\text{str}} G_{\text{str}} Q_{\text{str}}^T)^T
\]

and thus there exists a real orthogonal matrix \( T \) such that

\[
D := T^T \left( M_{\sigma,\text{as}} + Q_{\text{str}} G_{\text{str}} Q_{\text{str}}^T \right) T
\]

is a diagonal matrix. This diagonalized mass matrix \( D \) is still singular due to non-conducting regions and this proves that the mass matrix has a nilpotency of \( \nu = 1 \) because the Jordan normal form has no non-trivial blocks of type (B.13).

### 4.2.4 Semi-Explicit Formulation

We shall derive a decomposition of system (4.37) in its differential and algebraic parts; therefore we transform the first equation (4.37a) into its Kronecker normal form (KNF). Kronecker’s Theorem B.2 is applicable because the regularity of the matrix pencil was shown above and thus all prerequisites are fulfilled. Furthermore the DAE-index equals 1 in this particular case and thus the KNF yields a semi-explicit form.

Let the regular matrices \( P \) and \( Q \) transform the equation (4.37a) into its KNF, then the following decomposition

\[
P = \begin{pmatrix} P_y \\ P_z \end{pmatrix}, \quad Q = (Q_y, Q_z)
\]

can be defined, see equation (B.14) in the Appendix for the details. Utilizing these projectors we obtain the following reformulation of the field equation

\[
\frac{d}{dt} y_{\text{fit}}(t) + W_{\nu} y_{\text{fit}}(t) = P_y \tilde{j}_{\text{src}}, \quad (4.39a)
\]
\[
z_{\text{fit}}(t) = P_z \tilde{j}_{\text{src}}, \quad (4.39b)
\]

with the MVP \( \bar{a} = Q_y y_{\text{fit}} + Q_z z_{\text{fit}} \) that is decomposed in algebraic and differential parts, the regular matrix \( W_{\nu} := P_y K_y Q_y \) and finally on the right-hand side the source current density is defined as \( \tilde{j}_{\text{src}} := M_{\sigma,\text{as}} Q_{\text{sol}} u_{\text{sol}} + Q_{\text{str}} G_{\text{str}} u_{\text{str}} \).
Let us have a closer look on the algebraic part of the MVP

\[ z_{\text{fit}}(t) = P_z \vec{j}_\text{src} = P_z (M_{\sigma,\text{as}}Q_{\text{sol}}u_{\text{sol}} + Q_{\text{str}}G_{\text{str}}u_{\text{str}}). \]

It follows from Kronecker’s Theorem for our index-1 system

\[
\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = P (M_{\sigma,\text{as}} + Q_{\text{str}}G_{\text{str}}Q_{\text{str}}^T) Q
\]

and a left multiplication with \((I \ 0)\) gives

\[
(I \ 0) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} Q^{-1} = (I \ 0) P (M_{\sigma,\text{as}} + Q_{\text{str}}G_{\text{str}}Q_{\text{str}}^T),
\]

which yields

\[
(0 \ 0) = P_z (M_{\sigma,\text{as}} + Q_{\text{str}}G_{\text{str}}Q_{\text{str}}^T)
\Rightarrow (0 \ 0) = P_z (M_{\sigma,\text{as}} + Q_{\text{str}}G_{\text{str}}Q_{\text{str}}^T) M_{\sigma,\text{as}}^+ M_{\sigma,\text{as}}
\Rightarrow (0 \ 0) = P_z M_{\sigma,\text{as}},
\]

the last deduction holds because the property (4.28) guarantees that the second term vanishes

\[
Q_{\text{str}}^T M_{\sigma,\text{as}}^+ M_{\sigma,\text{as}} = Q_{\text{str}}^T M_{\sigma,\text{as}} M_{\sigma,\text{as}}^+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

On the other hand follows from above

\[
P_z Q_{\text{str}}G_{\text{str}}Q_{\text{str}}^T = -P_z M_{\sigma,\text{as}} = (0 \ 0)
\]

and therefore

\[
P_z Q_{\text{str}}G_{\text{str}} = P_z Q_{\text{str}}G_{\text{str}} Q_{\text{str}}^T Q_{\text{str}}^T (Q_{\text{str}}^T Q_{\text{str}})^{-1} = 0.
\]

We conclude that the voltages through solid and stranded conductors do not contribute to the algebraic part of the MVP, which therefore vanishes at all

\[ z_{\text{fit}}(t) = P_z (M_{\sigma,\text{as}}Q_{\text{sol}}u_{\text{sol}} + Q_{\text{str}}G_{\text{str}}u_{\text{str}}) = 0. \] (4.40)

This is a reasonable result since the only regions to which a coupling with \(Q_{\text{sol}}\) and \(Q_{\text{str}}\) is established are those with non-zero terms in the matrix \(M_{\sigma,\text{as}} + Q_{\text{str}}G_{\text{str}}Q_{\text{str}}^T\). This affects also the excitation current density \(\vec{j}_\text{dc}\), which is calculated from \(u_{\text{sol}}\) and \(u_{\text{str}}\). They contribute therefore not to the algebraic part \(z_{\text{fit}}\) but only to the differential part \(a_y\) in the Kronecker decomposition. Based on
the KNF is not only a separation, it allows us to reduce the MQS curl-curl equation and its unknown MVP. This is coarsely related to the common shrinking approach in traditional FIT solution algorithms, which removes specific lines before the solving and adds zeros into the solution afterwards. In our case the inflation and transformation of the solution vector to its original dimension is reached with the help of \( Q_y \), it now holds

\[
\bar{a} = Q_y y_{\text{fit}}. \tag{4.41}
\]

Inserting the above KNF (4.39) into the coupled system (4.37) yields finally the following semi-explicit coupling formulation

\[
\frac{d}{dt} y_{\text{fit}} + W_y y_{\text{fit}} = P_y \left( M_{\sigma,\text{as}} Q_{\text{sol}} u_{\text{sol}} + Q_{\text{str}} G_{\text{str}} u_{\text{str}} \right), \tag{4.42a}
\]

\[
\tau_{\text{sol}} = Q_{\text{sol}}^T K_y Q_y y_{\text{fit}}, \tag{4.42b}
\]

\[
\tau_{\text{str}} = G_{\text{str}} Q_{\text{str}}^T M_{\sigma,\text{aniso}}^+ K_y Q_y y_{\text{fit}}. \tag{4.42c}
\]

Since the algebraic part vanishes only the differential part (4.42a) is needed to describe the MVP and hence the magnetic field. Let us manifest this result in the following theorem

**Theorem 4.1.** If the matrix pencil \([M_{\sigma,\text{as}}, K_y]\) is regular (\(G1\)) and the discretization grid is sufficiently fine (\(G2\)), then the MQS curl-curl equation (4.37a) for solid and stranded conductors has an equivalent ODE (4.42a).

Note that we did not use any particular gauging approach to obtain the regular matrix pencil. The only assumption was that the used approach does not alter the discrete matrix of conductivities \(M_{\sigma,\text{as}}\) and that it preserves the symmetry and positive (semi-)definiteness of \(K_y\) alias \(\tilde{C} M_y C\).

A result that is similar to the Theorem 4.1 from above follows trivially for any approach that regularizes the matrix \(M_{\sigma,\text{as}}\) as for example the \(\kappa\)-regularization, since they produce (stiff) ODEs by construction.

### 4.2.5 Sparsity

The various reformulations were convenient for our theoretical analysis, but may become disadvantageous in numerical simulations, since the transformations destroy the symmetrical and sparse structure of the matrices involved. Let us discuss an example to demonstrate this loss. We start with the first voltage-dependent formulation (4.21) and multiply the system from the left by the matrix
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\( R_2 \), this yields the matrix equation

\[
\begin{pmatrix}
M_{\sigma,as} + Q_{str} G_{str} Q_{str}^T & 0 & 0 \\
Q_{sol}^T M_{\sigma,as} & 0 & 0 \\
Q_{str}^T & 0 & 0
\end{pmatrix}
\dot{x}_{\text{fit}} + \begin{pmatrix}
K_{\nu} & 0 & 0 \\
0 & I & 0 \\
0 & 0 & R_{str}
\end{pmatrix} x_{\text{fit}} = \begin{pmatrix}
M_{\sigma,as} Q_{sol} u_{sol} + Q_{str} G_{str} u_{str} \\
0 \\
G_{sol} u_{sol}
\end{pmatrix}, \quad (4.43)
\]

with \( x_{\text{fit}} := (\bar{a}, t_{sol}, t_{str})^T \). The multiplication affects only the first line, but this is the critical one since it contains all the equations from the discretization process of the FIT while the other two coupling lines are only as many equations as conductors are present in the circuit (\( N_{sol} + N_{str} \)). The second multiplication by matrix \( R_1 \) that would yield the final system (4.37) is not discussed here since it has a minor effect on the system.

The transformation yields an analytically equivalent system since the multiplication corresponds to a simple block-Gauß elimination but on the other hand it is numerically cumbersome because the diagonal (sparse) mass matrix \( M_{\sigma,as} \) is replaced by a non-diagonal and less sparse term

\[
M_{\sigma,\text{fillin}} := M_{\sigma,as} + Q_{str} G_{str} Q_{str}^T. \quad (4.44)
\]

The result is obviously a huge number of fill-ins. Fig. 4.8 gives an exemplary plot of both matrices; it shows the structure of the diagonal conductivity matrix and its non-diagonal counterpart. The numerical solution of the denser system will
come with higher computational costs since an (iterative) solver has to perform more basic linear algebra operations in each step, but the denser system might need less iterations as observed by De Gersem and Weiland [21].

## 4.3 Summary

In this chapter two conductor models were derived that may be coupled to electric circuits. Arbitrary many of those conductors can be included in one field model and the resulting system remains an index-1 DAE. Analytically the situation is even better, since we have shown that the algebraic part of the MQS curl-curl equation vanishes and thus we have an ODE.

In the next chapter we shall establish a coupling between the FIT model and the circuit described by MNA. We discuss the case of a coupling to one field model (including several conductors) for simplicity purposes only, although a generalization is straight-forward.
Chapter 5

FIT/MNA Coupling

5.1 Problem Formulation

The FIT/MNA coupled problem includes two sub-problems, the electric network (MNA) and the electromagnetic field system (FIT). They are coupled by an interface, that was modelled in Section 4.1 for two conductor types, the solid and stranded conductor. The formulations fit in the MNA framework, because they describe current-voltage relations that are similar to the characteristic equations in traditional circuit simulations.

The coupling requires us to assign field variables to circuit ones and vice versa. Therefore the branch voltages $u_{\text{sol}}$ and $u_{\text{str}}$ of the FIT model (4.37) are translated into differences of node potentials in the MNA. This is done by utilizing two incidence matrices $A_{\text{sol}}$ and $A_{\text{str}}$ that are constructed as defined in equation (3.1).

The translation reads

$$u_{\text{sol}} = A_{\text{sol}}^T e,$$

$$u_{\text{str}} = A_{\text{str}}^T e,$$

(5.1)
(5.2)

where $e$ denotes the vector of node potentials as introduced for the MNA.

Those two definitions result directly in a first formulation of the coupled problem. We only need to augment the systems of the MNA (3.15) and FIT (4.20) sub-problems to a comprehensive problem formulation and replace $\lambda$ with the conductor currents $i_{\text{sol}}$ and $i_{\text{str}}$ in the current balance of the MNA system. The following arrangement is obtained, when the system is sorted into electric network, coupling and field equations.
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electric network:

\[ A_C \frac{d}{dt} q + A_R r(A_T e, t) + A_L j_L + A_V j_V + A_I(t) + A_{sol} t_{sol} + A_{str} t_{str} = 0, \quad (5.3a) \]
\[ \frac{d}{dt} \Phi - A_L^T e = 0, \quad (5.3b) \]
\[ A_V^T e - v(t) = 0, \quad (5.3c) \]
\[ q - q_C(A_C e, t) = 0, \quad (5.3d) \]
\[ \Phi - \Phi_L(t_L, t) = 0, \quad (5.3e) \]

coupling interface:

\[ \tilde{j}_{src} - M_{\sigma,as} Q_{sol} A_{sol}^T e - Q_{str} t_{str} = 0, \quad (5.3f) \]
\[ t_{sol} + Q_{sol}^T M_{\sigma,as} \frac{d}{dt} \tilde{a} - G_{sol} A_{sol}^T e = 0, \quad (5.3g) \]
\[ R_{str} t_{str} + Q_{str}^T \frac{d}{dt} \tilde{a} - A_{str}^T e = 0, \quad (5.3h) \]

electromagnetic field:

\[ M_{\sigma,as} \frac{d}{dt} \tilde{a} + K_{\nu} \tilde{a} - \tilde{j}_{src} = 0. \quad (5.3i) \]

The above problem formulation (5.3) is very close to the well-known formulations of the FIT and MNA problems, but anyhow it is not satisfying because of various reasons. One reason is that the formulation as n-terminal device would benefit from a unified notation of both conductor models, but the solid and stranded cases are still treated separately. Furthermore the coupling terms include the derivative of the MVP, this is not convenient either. Therefore an equivalent formulation is proposed in Box 5.1. This is obtained by augmenting the two incidence matrices to \( A_\lambda = (A_{sol}, A_{str}) \) and yields

\[ A_\lambda^T e = u_{fit} = \begin{pmatrix} u_{sol} \\ u_{str} \end{pmatrix}. \quad (5.4) \]

Its application to the current density \( \tilde{j}_{src} \) within equation (4.37) gives

\[ \tilde{j}_{src} = M_{\sigma,as} Q_{sol} u_{sol} + Q_{str} G_{str} u_{str} \]
\[ = (M_{\sigma,as} Q_{sol}, Q_{str} G_{str}) \begin{pmatrix} u_{sol} \\ u_{str} \end{pmatrix} \]
\[ = (M_{\sigma,as} Q_{sol}, Q_{str} G_{str}) A_\lambda^T e. \quad (5.5) \]
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Box 5.1: Unified FIT/MNA Coupled System

electric network:
\[ A_C \frac{d}{dt} q + A_R r(A_T e, t) + A_L \dot{J}_L + A_V \dot{J}_V + A_I(t) + A_\lambda \lambda = 0, \]  \hspace{1cm} (5.8a)
\[ \frac{d}{dt} \Phi - A_T^T e = 0, \]  \hspace{1cm} (5.8b)
\[ A_T^T e - v(t) = 0, \]  \hspace{1cm} (5.8c)
\[ q - q_C(A_C e, t) = 0, \]  \hspace{1cm} (5.8d)
\[ \Phi - \Phi_L(\dot{u}_L, t) = 0, \]  \hspace{1cm} (5.8e)

coupling interface:
\[ \frac{\vec{j}_{\text{src}}}{H_\sigma} - H_\sigma A_\lambda^T e = 0, \]  \hspace{1cm} (5.8f)
\[ \lambda - H_\nu \vec{a} = 0, \]  \hspace{1cm} (5.8g)

electromagnetic field:
\[ (M_{\sigma, \text{as}} + Q_{\text{str}} G_{\text{str}} Q_{\text{str}}^T) \frac{d}{dt} \vec{a} + K_{\nu} \vec{a} - \frac{\vec{j}_{\text{src}}}{H_\nu} = 0. \]  \hspace{1cm} (5.8h)

This gives equation [5.8i] of the coupling interface in Box 5.1. Furthermore the currents \( \dot{J}_{\text{sol}} \) and \( \dot{J}_{\text{str}} \) of the conductor models are also assembled to a single quantity
\[ \lambda = \begin{pmatrix} \dot{J}_{\text{sol}} \\ \dot{J}_{\text{str}} \end{pmatrix}, \]  \hspace{1cm} (5.6)
which contributes to the current balance of the electric network and allows a reformulation of the field system. The conductor currents defined in equations [4.37b] and [4.37c] now give the second equation of the coupling interface [5.8g] of Box 5.1.

\[ \lambda = \begin{pmatrix} \dot{J}_{\text{sol}} \\ \dot{J}_{\text{str}} \end{pmatrix} = \begin{pmatrix} Q_{\text{sol}}^T K_{\nu} \vec{a} \\ G_{\text{str}} Q_{\text{str}}^T M_{\nu, \text{aniso}}^+ K_{\nu} \vec{a} \end{pmatrix} = \begin{pmatrix} Q_{\text{sol}} M_{\nu, \text{aniso}}^+ Q_{\text{str}} G_{\text{str}}^T \end{pmatrix} K_{\nu} \vec{a}. \]  \hspace{1cm} (5.7)

Let us summarize the contributions to Box 5.1, we have one linear equation for each conductor device, solid and stranded respectively. The corresponding equation [5.8g] was derived above from the system of conductor models [4.37]. This is also the origin of the definition of the direct current density \( \vec{j}_{\text{src}} \) in equation [5.8f]. Those two lines form the coupling interface, since they include variables of
the electric circuit and the electromagnetic field problem. The latter is described in terms of the MVP by the differential-algebraic equation \((5.8h)\).

Equation \((5.8h)\) is the largest and computationally most expensive part of the coupled system since it is the result of the FIT discretization and thus consists of a comparatively huge number of equations. Finally the remaining equations \((5.8a)–(5.8e)\) in Box 5.1 correspond exactly to those lines of the flux-/charge-oriented MNA system \((3.15)\).

### 5.2 Numerical Analysis

Let us assume that the MVP is regularized by a gauge and thus the MQS curl-curl system has a regular matrix pencil as stated in condition \(G1\). Let furthermore the discretization grid be sufficiently fine to ensure the correct resolution of the conductor models, which is condition \(G2\). Those presumptions guarantee the existence of the Kronecker normal form and the semi-explicit formulation of the MQS field system \((1.39)\). On the other hand we postulate that the network fulfills the topological conditions \(T1\) and \(T2\), since this gives us another semi-explicit formulation, this time for the MNA system \((3.20)\). We will see that both semi-explicit formulations are fundamental for our analysis.

#### 5.2.1 Linear Problem

We restrict our analysis to the linear case of Box 5.1, although a generalization to the non-linear case is straight forward. The resulting loss of generality concerns mainly the electrical network, since non-linear field problems were not introduced so far. However linearization is a common approach for analyzing general circuits and this is also true for field problems. Hence we discuss the linear system \((5.9)\) that arises either because of linear materials/elements or as an approximation.

\[
A_CCA^T_d e + A_RG A^T_R e + A_{LJL} + A_{VJV} + A_I + A_\lambda \lambda = 0, \quad (5.9a)
\]
\[
L_{JL} - A^T_L e = 0, \quad (5.9b)
\]
\[
v - A^T_V e = 0, \quad (5.9c)
\]
\[
\jmath_{src} - H_\sigma A^T_\lambda e = 0, \quad (5.9d)
\]
\[
\lambda - H_\nu \tilde{a} = 0, \quad (5.9e)
\]
\[
(M_{\sigma,as} + Q_{str}G_{str}Q^T_{str}) \frac{d}{dt} \tilde{a} + K_\nu \tilde{a} - \jmath_{src} = 0 \quad (5.9f)
\]
5.2.2 Monolithic Approach

We start with the analysis of the numerical properties of the monolithic approach to the coupled problem (Box 5.1), in other words we study the case in which both sub-problems are solved simultaneously within one system. We will proceed analogously to the technique in the corresponding section on the numerical properties of the linear network, see Section 3.3.1. We have already developed a linear formulation (5.9) of the coupled problem (Box 5.1) and shall separate the algebraic from the differential parts in the following. Afterwards we analyze the resulting matrices to conclude results regarding uniqueness and index.

The decomposition of the field variables was studied before and so we can take advantage of the semi-explicit system (4.42). We have to replace the source current density \( \vec{j}_{\text{src}} \) with the new definition \( H_\sigma A^T_e \) from above to obtain a field problem that is formulated in terms of the MNA node potentials

\[
\begin{align*}
\frac{d}{dt} y_{\text{fit}} + W_\nu y_{\text{fit}} = & \mathcal{P}_y H_\sigma A^T_e e \\
z_{\text{fit}} = & 0.
\end{align*}
\] (5.10a)

The algebraic part vanishes according to equation (4.40), so the remaining ODE (5.10a) replaces the implicit differential-algebraic field equation (5.9f). In addition the coupling equation (5.9d) is not needed anymore because we inserted it and brought it to the right-hand side of the equation (5.10a).

When considering the semi-explicit network formulation (3.20), one can obviously add (5.10a) to the differential equations originating from the network. This yields

\[
\begin{align*}
\begin{pmatrix} H_C & 0 & 0 \\
0 & L & 0 \\
0 & 0 & I \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \mathcal{P}_C e \\
\mathcal{J}_L \\
y_{\text{fit}} \end{pmatrix} + \begin{pmatrix} A_R G A^T_L & A_L & 0 \\
-A^T_L & 0 & 0 \\
-P_y H_\sigma A^T_\lambda & 0 & W_\nu \end{pmatrix} \begin{pmatrix} \mathcal{P}_C e \\
\mathcal{J}_L \\
y_{\text{fit}} \end{pmatrix}
+ \begin{pmatrix} A_R G A^T_R & A_V & A_\lambda \\
-A^T_L & 0 & 0 \\
-P_y H_\sigma A^T_\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{Q}_C e \\
\mathcal{J}_V \\
\lambda \end{pmatrix} + \begin{pmatrix} A_I \bar{y} \\
0 \\
0 \end{pmatrix} = 0.
\end{align*}
\] (5.11a)

The last missing piece is the current \( \lambda \) of the coupling interface (5.9e), which reads in terms of the MVP

\[
\begin{align*}
0 = & \lambda - H_\nu \bar{a} \\
= & \lambda - H_\nu Q_y y_{\text{fit}}
\end{align*}
\] (5.11b)

This equation can either be inserted in the differential part (5.11a) to eliminate the dependence on \( \lambda \) or equivalently \( \lambda \) can be treated as an additional algebraic
variable. We prefer the latter formulation because it keeps both sub-problems separate. The algebraic part reads

\[
\begin{pmatrix}
Q^T R G A^T R Q & Q^T C A v & Q^T C A \lambda \\
A^T V Q C & 0 & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
Q C e \\
\nu \\
\lambda
\end{pmatrix}
+ \begin{pmatrix}
Q^T A R G A^T R P C & Q^T C A L \\
-A^T V P C & 0 & 0 \\
0 & 0 & -H\nu Q y
\end{pmatrix}
\begin{pmatrix}
P C e \\
J L \\
J y_{fit}
\end{pmatrix}
+ \begin{pmatrix}
Q^T A f \\
-v \\
0
\end{pmatrix} = 0.
\] (5.11b)

The regular matrix \( H_C = A_C C A^T C + Q^T C Q \) and both matrices \( H_{\sigma} = (M_{\sigma, as} Q_{sol} + Q_{str} G_{str}) \) and \( H_{\nu} = (Q_{sol}, M_{\sigma, aniso} Q_{str} G_{str})^T K_{\nu} \) were defined before for notation purposes.

Now, having derived the linear monolithic coupled system (5.11), we can analyze its properties. The following theorem guarantees uniqueness and index-1 with similar arguments as for the MNA-only system.

**Theorem 5.1.** If the MNA model is linear and the topological conditions \( T_1, T_2 \) hold and if the FIT model fulfills the conditions \( G_1, G_2 \), then the monolithic FIT/MNA coupled model (Box 5.1) leads to an index-1 DAE.

**Proof.** The proof is simple because the work has been done in the derivation of both semi-explicit formulations. Now all we have to do is to show the regularity of the matrix

\[
B_{\text{mono}} := \begin{pmatrix}
Q^T A R G A^T R Q & Q^T C A v & Q^T C A \lambda \\
A^T V Q C & 0 & 0 \\
0 & 0 & I
\end{pmatrix},
\] (5.12)

which is trivial, since the topological conditions \( T_1 \) and \( T_2 \) guarantee

\[
\det (B_{\text{mona}}) = \det \begin{pmatrix}
Q^T A R G A^T R Q & Q^T C A v \\
A^T V Q C & 0
\end{pmatrix} \neq 0.
\]

The proof is finally completed by analogous arguments as for Theorem 3.1.

### 5.2.3 Co-Simulation

There are already highly advanced simulators available for both circuit and field problems. It is obviously advantageous to use those packages to solve the sub-problems, since they are tailored for the particular case. If they are solved independently of each other this approach does not only take advantage of the multi-method but also of the multi-rate techniques, since it makes different simulators
and time steps possible. The idea is called *co-simulation* (simulator coupling) and we have to study the properties of a *dynamic iteration* or *waveform relaxation* scheme, see Burrage \[13\].

It is crucial to keep the FIT and MNA sub-problems separated to take advantage of the dynamic iteration approach. We consider therefore the linear systems (3.17) and (4.21), which can be solved by circuit and field simulators, respectively. Both are systems of DAEs, whose particular right-hand side depends on the other system

\[
M_{\text{mna}} \cdot \dot{x}_{\text{mna}}(t) + K_{\text{mna}} \cdot x_{\text{mna}}(t) = k(t, \lambda),
\]

(5.13a)

\[
M_{\text{fit}} \cdot \dot{x}_{\text{fit}}(t) + K_{\text{fit}} \cdot x_{\text{fit}}(t) = k(u_{\text{fit}}),
\]

(5.13b)

since \( \lambda = (i_{\text{sol}}, i_{\text{str}})^T \) equals the current components of \( x_{\text{fit}} := (\bar{a}, i_{\text{sol}}, i_{\text{str}})^T \) and \( u_{\text{fit}} = A_{\lambda}^T e \) is derived from the node potentials of \( x_{\text{mna}} := (e, j_L, j_V)^T \).

Let us give a more abstract description of system (5.13) that utilizes the semi-explicit formulations of the field and network problem respectively. We follow the notation of Bartel \[5\], but note that this formulation does not reflect the linearity of the sub-problems. The FIT/MNA coupling reads

\[
\dot{y}_{1a} = f_{1a}(y_{1a}, z_{1a}, z_{2b}) \quad y_{1a}(0) = y_{1a,0} \quad y_{1a} = (P_C e, j_L)^T \quad z_{1a} = (Q_C e, j_V)^T
\]

(5.14a)

and

\[
\begin{align*}
\dot{y}_{2a} &= f_{2a}(y_{2a}, z_{2a}) \\
0 &= h_{1a}(y_{1a}, z_{1a}) \\
0 &= z_{2a} - h_{2a}(y_{1a}, z_{1a}) \\
0 &= z_{2b} - h_{2b}(y_{2a}) \\
y_{2a}(0) &= y_{2a,0} \quad y_{2a} = y_{\text{fit}} \\
z_{2a} &= \tilde{j}_{dc} \\
z_{2b} &= \lambda
\end{align*}
\]

(5.14b)

provided that the topological conditions in MNA, the gauge and grid conditions are fulfilled. The MQS curl-curl equation is represented by an ODE in (5.14b) according to Theorem \[4\] and in addition the supporting functions, namely the source current density \( (z_{2a} = \tilde{j}_{dc}) \) and the derived currents \( (z_{2b} = \lambda) \) are evaluations rather than algebraic equations and thus both are dispensable. This is important, since analytically the coupled problem becomes a DAE-ODE system as in Bartel \[5\].

**Gauss-Seidel Approach**

The dynamic iteration requires us to subdivide the time domain into a coarse grid of communication times \( 0 = H_0, H_1, \ldots, H_{n_H} = \tau \). The interval between two communication times \( H_i \) and \( H_{i+1} \) with \( i \in \{0, \ldots, (n_H - 1)\} \) is called \((i-\text{th})\)
Figure 5.1: Dynamic Iteration. The first time window exhibits solid, the following dashed arrows; macro steps are represented by red coloured arrows, while micro steps are blue; the grey arrow indicates that a single time window might be calculated several times (iteratively).

time window. Let us assume an equidistant grid with time windows of length $H_{i+1} - H_i = H$.

We still assume that the electric circuit varies faster than the magnetic field because of the magnetoquasistatic assumption (we will reinspect this assumption in the next chapter). Thus the field can be integrated using only one (macro) step of length $H$, while the network might be integrated with multiple (micro) steps at the same time. The computation of the window is furthermore repeated iteratively see Fig. 5.1. Let $l_i$ denote the number of iterations that depends on the particular window.

We shall analyze in the following the time first window $[0, H]$ only; the error transport is disregarded for the moment, but it is studied for example in Arnold and Günther [2]. The window is iterated $l_0$-times and this yields the approximate solution

\[
(\tilde{y}, \tilde{z})|_{[0, H]} = (y_0^0, z_0^0)|_{[0, H]},
\]

where the iterated results $y^{(k)} := (y_{1a}^{(k)}, y_{2a}^{(k)})^T$ and $z^{(k)} = z_{1a}^{(k)}$ with $(k \geq 1)$ are computed by a Gauß-Seidel waveform relaxation algorithm. Alternatively a Jacobi type scheme with the advantage of higher computational parallelism could have been chosen, see [13].

We start with the integration of the presumably faster varying system which is the circuit DAE-IVP in our case because of the MQS assumption. The MNA system depends on data from the second system, which is the field DAE-IVP obtained from FIT. These missing data, i.e., the currents $\lambda$ flowing through the conductors, are unknown at the current time. Hence we extrapolate the old iterates to the current time and use them as new initial values. We choose the following constant extrapolation for simplicity reasons

\[
y_{2a}^{(0)} := y_{2a,0}.
\]
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Providing this data the first system can be solved

\[
\begin{align*}
\dot{y}_{1a}^{(k)} &= f_{1a}(y_{1a}^{(k)}, z_{1a}^{(k)}, y_{2a}^{(k-1)}) \\
0 &= h_{1a}(y_{1a}^{(k)}, z_{1a}^{(k)}), \\
y_{1a}^{(k)}(0) &= y_{1a,0}
\end{align*}
\]

(5.17a)

and in the second step the field DAE-IVP is integrated using the data that was obtained from the circuit integration before

\[
\begin{align*}
\dot{y}_{2a}^{(k)} &= f_{2a}(y_{2a}^{(k)}, y_{1a}^{(k)}, z_{1a}^{(k)}) \\
y_{2a}^{(k)}(0) &= y_{2a,0}.
\end{align*}
\]

(5.17b)

Note that the evaluations \(z_{2a} = \vec{\psi}_{dc}\) and \(z_{2b} = \lambda\) are inserted to clarify the general structure, even if this results in a misuse of the introduced notation.

System (5.17) exhibits a similar structure to other multi-physical problems and in particular to the thermal/electrical problem studied by Bartel [5]. Thus we can apply the theory given there. Especially Corollary 8.1 is applicable, since no algebraic variable depends on an old iterate. We could additionally assume that the topological condition \(H1\) holds, if we want to simplify the structure further.

In this simplified case the coupling is established only by differential variables, since the second equation (5.17b) is only controlled by the capacitive branches \(y_{1a}\) and the dependence on \(z_{1a}\) vanishes. However in both cases the convergence is guaranteed because the contraction is achieved with a vanishing factor \(\alpha = 0\) as it is called by Bartel [5].

Note that this result is not directly transferable to a dynamic iteration approach that uses the original mixed voltage- and current-driven formulation (4.20). On the one hand the MNA sub-problem will require additional algebraic equations that correspond to current dependent voltage sources and on the other hand the semi-explicit formulation of the field is not applicable.

Multi-Rate Co-Simulation

We want to introduce a special case of the above Gauss-Seidel scheme, that is referred in [5] to as full multi-rate co-simulation. This approach does not belong to the class of waveform-relaxation schemes anymore. It needs additional investigations, since it iterates each time-window only once, in other words \(l_i = 0\) for \(i = 0, \ldots, (n_H - 1)\). We shall regard only numerically experiments of this variation, see the next section for the implementational details of this and its waveform counterpart.
5.3 Implementation

In this chapter we want to give a brief overview on the algorithms that shall be used in this treatise to solve FIT, MNA and coupled problems. We will not go into any numerical or programming details.

The plain MNA problems are solved with MATLAB’s built-in ODE suite. There are two solvers that are capable of solving DAEs: the first is ode15s, which is based on the numerical differentiation formulas (NDF) or alternatively the backward differentiation formulas (BDF). The second possibility is ode23t, which uses a variant of the trapezoidal rule. We refer for a general overview on the algorithms of the ODE suite to Shampine and Reichelt [52] and for the solution of index-1 DAEs with MATLAB to Shampine et al. [51].

Our software package “FIDES” (Finite Integration DEvice Simulator) is implemented in MATLAB and supports the OCTAVE platform with some reservations; it is freely redistributable software and may be redistributed and/or modified under the terms of the GNU General Public License. Note that the headers of the main m-files are added in the Appendix C.

5.3.1 FIT and Monolithic Coupled Problems

Let the ODE-IVP be given as introduced in the Appendix B.1, i.e.,

\[ \dot{y}(t) = f(t, y(t)), \quad y(0) = y_0 \]

that describes an unknown function \( y \) on a time interval \([0, \tau]\) of interest. We discretize this interval using a (not necessarily constant) grid so that \( 0 = t_0 < t_1 < \ldots < t_n = \tau \) holds with step sizes \( h_i := t_{i+1} - t_i \). Let us denote the numerical approximations of \( y(t_i) \) by \( y_i \).

Euler Method

The simplest method to solve the above ODE-IVP was derived by Euler and first published in 1768. The derivative \( \dot{y}(t) \) is replaced by a difference quotient in forward direction and hence the method is called forward or explicit Euler method

\[ y_{i+1} = y_i + h_i f(t_i, y_i) \] (5.18)

---

1Matlab is a trademark of The MathWorks, Inc., see [http://www.mathworks.com/](http://www.mathworks.com/)

2Octave is a MATLAB-compatible programming language, see [http://www.octave.org](http://www.octave.org)
and analogously the \textit{backward} or \textit{implicit Euler method} is derived, when the derivative is replaced by a backward difference quotient
\begin{equation}
y_{i+1} = y_i + h_i f(t_{i+1}, y_{i+1}).
\end{equation}

The problems that are addressed in this treatise are linear DAEs of the type \eqref{eq:dae} rather than an ODE and thus the IVP reads
\begin{equation}
M \cdot \dot{x}(t) + K \cdot x(t) = k(t), \quad x(0) = x_0,
\end{equation}
with a singular constant coefficient matrix $M$, as introduced in the Appendix \ref{ap:dae}. Note that the unknown is now called $x$ to express the differential-algebraic character. Solving the DAE-IVP with implicit Euler method yields the following linear system of equations in every integration step
\begin{equation}
(K + \alpha M) x_{i+1} = \alpha M x_i + k(t_{i+1}),
\end{equation}
where $\alpha = 1/h$ is introduced to resemble the definition of the matrix pencil $[M, K]$, see Definition \ref{def:matrix_pencil}. The demonstration code for the FIT and the FIT/MNA coupled problems (see Listing \ref{lst:fit_mna}) bases on the implicit Euler method as described by system \eqref{eq:dae}. It is a good choice because of its straightforwardness and furthermore its numerical robustness, although it is only a method of order 1. We shall neither prove any of those numerical properties nor will we introduce the corresponding concepts that are needed to justify the choice. We refer for stiffness, order and stability of implicit Runge-Kutta methods in general and of the implicit Euler method in particular to the common literature, see for example Deuflhard et al. \cite{deuflhard2002}.

There already exist superior algorithms to the FIT problem that use higher order implicit Runge-Kutta methods. An implementation is given in \cite{clemens2001} and some more details are discussed in \cite{kaasschieter2001}.

\section*{Iterative Solution of the Linear System}

Integrating FIT problems with the implicit Euler method yields the linear system \eqref{eq:dae}. In this case the matrix $M$ equals the discrete matrix of conductivities $M_\sigma$ and $K$ is the curl-curl matrix $K_\nu$. Both matrices are comparatively large, but sparse and their matrix pencil $[M, K]$ is generally singular, if a FIT problem without a gauging technique is regarded. In this case the system has obviously a rank deficiency. In spite of the singularity the system can be solved with a Krylov subspace method, as analyzed generally by Kaasschieter \cite{kaasschieter2001} and specifically for FIT by Clemens et al. \cite{clemens2001}.
Before the linear system is actually numerically solved the matrices are *shrunk*, that means negligible zero-lines are removed from the system of equations. Note that this reduction is no regularization and the problem remains singular. Fig. 5.2 shows the matrix pencil before and after the shrinking process. The Krylov solver is then applied afterwards and finally the corresponding zero entries are reinserted in the solution vector. This process leads to equivalent results, but has better convergence properties.

We use *MATLAB*’s build-in implementation of the Preconditioned Conjugate Gradient method (PCG) to solve the curl-curl equation and the Biconjugate Gradient method (BICG) for the coupled problems to avoid difficulties due to asymmetry, see Barrett et al. [4] and Saad [49]. The required tolerance of the residual is chosen quite strictly, since its significance for the error decreases if the condition number is bad. In our case of a singular system we refer here to the *effective spectral condition number* which is the quotient of the maximum and minimum non-zero eigenvalue [38, 49]. In fact the condition number will become critical, especially when a gauging is applied. We set the tolerance to a value of

\[ PCGtol := 1 \cdot 10^{-8}. \] (5.21)

**Adaptive Step Size Control**

In practice proper step sizes \( h_i \) are a priori unknown and too large choices yield a corresponding large discretization error, while too small choices result in inefficiently high computational costs. There comes the idea of an *adaptive step*
size control into play. The algorithm should adaptively adjust the step size itself depending on the problem. The general idea is to integrate one time step of the problem with an estimated step size using an method with a particular accuracy and compare it with a solution of higher accuracy. Then the difference between both solutions is calculated. This is not the real error as a matter of fact since the exact solution is not available. However one can derive estimates and prove their reliability, see Stoer and Bulirsch [55], Section 7.2.5.

We choose the basic algorithm from Fig. 5.4 for our implementation backeuler, see Listing C.5 for the MATLAB header. It uses two additional integration steps with a step size that is divided in halves instead of a higher order method. We choose the error norm that is suggested by Hairer et al. [34]. It follows for integration schemes of order 1

\[
\epsilon_i = Atol + \max(|y_{0,i}|, |y_i|) \cdot Rtol, \quad (5.22)
\]

\[
err = \sqrt{\frac{1}{n} \sum_i \left( \frac{\dot{y}_i - y_i}{\epsilon_i} \right)^2}, \quad (5.23)
\]

where \(i \in \{1, \ldots, n\}\) and \(n\) denotes the number of unknowns. The corresponding step size prediction reads

\[
h_{new} = \min \left( \alpha \cdot h_0, \max \left( \beta \cdot h_0, \sqrt{\frac{\rho}{err}} \cdot h_0 \right) \right) \quad (5.24)
\]

with given absolute and relative tolerances \(Atol\) and \(Rtol\), respectively, a safety parameter \(\rho\) and finally \(\alpha\) and \(\beta\) that avoid “zigzagging”. This and other implementational details can be found in Hairer et al. [34], II.4. The tolerances are set in our implementation to

\[
Atol = 1 \cdot 10^{-3}, \quad Rtol = 1 \cdot 10^{-6} \quad (5.25)
\]

and the general parameters are chosen as follows

\[
\rho = 0.9, \quad \alpha = 1.5, \quad \beta = 0.5. \quad (5.26)
\]

Please note that it might be problematic to calculate the error of the whole solution vector, when solving a FIT related problem due to the non uniqueness of the MVP. Two magnetic vector potentials might differ but describe the same magnetic field, see Definition 2.1. Although such a behaviour is not expected because of the properties of the Krylov subspace methods. On the other hand one can use the derived, but unique quantities (currents/voltage drops) when solving a coupled system.
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Figure 5.3: Step Size Clustering. Each cluster is an interval around a preferred step size $h_{\text{pref}}$; the last cluster limits the maximal increase to $\beta = 2$

### Step Size Clustering

We modified the general algorithm to obtain a more problem-oriented version. On one hand increases of the step size after a reject are forbidden as suggested by Shampine and Watts [53] and on the other hand a clustering is introduced that prevents a recalculation of the matrix pencils. This approach is similar to the idea of retaining the step size to save LU-decompositions as suggested in Hairer et al. [33], IV.8.

The error control suggests for every step a new step size $h_{\text{new}}$ and thus the matrix pencils for both integrations with full and half step size are recalculated as well. This is not efficient since $h_{\text{new}}$ may only marginally differ from the previous step size $h_0$ and the pencils could be reused. This becomes even more considerable, if we think of the FIT case, in which the boundary conditions and the shrinking are applied after the calculation of the matrix pencil. This idea can be developed even further: in the case of reducing the step size by half, the matrix pencil and in addition the result of the previous calculation $\hat{x}_0$ are already available and can be exploited for $y_1$. Finally in the case of doubling the step size the old matrix pencil for $x_0$ is utilizable for the calculation of $\hat{x}$.

We implemented a simple clustering on bases of the previous step size control (backeuler, see Listing C.5). The clustered control sets a new step size $h_{\text{new}}$ to a preferred step size $h_{\text{pref}} \in \{h_0/2, h_0, 2h_0\}$ if it is in the neighbourhood of $h_{\text{pref}}$, see Fig. 5.3. On one hand this avoids computational costs but on the other hand the clustering will need more integration steps, because step size increases are not regarded, even though they are comparatively small.

#### 5.3.2 Co-Simulation

We implemented for the dynamic iteration approach (gaussseidel) an adaptive Gauß-Seidel type scheme in Listing C.6. It does not adaptively control the step sizes as the algorithm before, they are fixed to a constant macro step size $H$, but the number of iterations $l_i$ can be determined dynamically.

The integration of (5.17) is iterated $k = 1, \ldots, l_i$ times and each following integration of the circuit uses the previous data of the field for extrapolation. We
CHAPTER 5. FIT/MNA COUPLING

\[ t_0 = ? \]
\[ x_0 = ? \]
\[ h_0 = ? \]

solve system (5.20) for \( x \) with stepsize \( h = h_0 \)

solve system (5.20) for \( \hat{x} \) with two steps \( h = h_0/2 \)
calculate the error \( ERR \)
calculate step size \( h_{\text{new}} \)

Is tolerance fulfilled? \((ERR \leq 1)\)

yes

\[ t_0 := t_0 + h_0 \]
\[ x_0 := \hat{x} \]
\[ h_0 := h_{\text{new}} \]

end

no

repeat last step with \( h_0 := h_{\text{new}} \)

Figure 5.4: Adaptive Step Size Control Algorithm. Basic algorithm, this diagram is essentially taken from [55].
require at least two iterations and compare the increment of two successive solutions. This acts as a makeshift because we have no error estimate derived, although the energy conservation might by a good starting point. However, if the increment fulfills a given tolerance the integration is continued on the next time window, in the other case additional iteration are performed until the results meets the tolerance or a maximal number of iteration is reached ($l_{\text{max}}$). This is implemented using basically the same error norm as for the adaptive step size control of the Euler method.

The iterations $l_i$ are chosen only dynamically as explained above if the function \texttt{gaussseidel} is called with the parameter $l = -1$. Any larger integer fixes the number of iterations and in coherence with Section \ref{sec:5.2.3} the multi-rate co-simulation is obtained for $l = 0$. It still utilizes the same algorithm but without the iteration of a particular time step.

\section*{5.4 Summary}

In this chapter a coupling formulation of the FIT/MNA problem has been proposed and discussed. The properties of the monolithic and dynamic iterated approach were analyzed and both were feasible without the need of additional conditions. We have seen that one could overcautiously require the conductor models to be inserted at capacitive branches only (H1), which is a natural choice since they are special cases of voltage-controlled current sources to the MNA.

Two basic algorithms \texttt{backeuler} and \texttt{gaussseidel} for the monolithic and dynamic iteration were given with their particular parameters. They will be applied to concrete examples in the next chapter.
CHAPTER 5. FIT/MNA COUPLING

Figure 5.5: *Dynamic Iteration Algorithm.* Basic algorithm for the dynamic iteration
Chapter 6

Computational Examples

We shall prove the concept of our proposed coupling approach using some computational examples. We start with an introductory FIT-only problem, then we give a plain MNA and finally a coupled problem. All examples were calculated on an Apple MacBook, Intel Core 2 Duo, 2 Ghz, 3GB RAM running OS X 10.4.10 and MATLAB 2007a.

6.1 Transformer

6.1.1 Introduction

We start with a simple introductory example for the FIT. We have chosen a transformer because this electromagnetic device is on the one hand simple enough but on the other hand incorporates both conductor models, as we will see later. It will allow us to demonstrate the method within a realistic scenario.

A transformer transfers energy between circuits operating at a different voltage level through a shared magnetic field. This effect is commonly used to change the magnitude of an alternating voltage, since a changing current in one circuit creates a magnetic field through the connected coil, which induces a voltage in another coil. This results in a current flow, if a load is added. See Gerthsen and Meschede [31] for the physical background.

For simplicity we shall study the behaviour of a transformer with two coils (stranded conductors in terms of FIT) in this section. This configuration is called single-phase transformer. Each coil has a different numbers of windings which are wound around a common magnetic core (solid conductor). Fig. 6.1(a) shows a sketch with red colored coils and a grey core. The latter is laminated,
to reduce the loss due to eddy currents. It consequently avoids an overheating of the device [31].

In traditional network analysis a real transformer is modelled by an equivalent circuit whose parameters are fitted to meet the behaviour of the transformer. This circuit does not utilize the magnetic field and thus a simple FIT model with two stranded conductors is indeed an advance on those models. In this section the circuit reduces to a transformer connected to two voltage sources $v_1(t)$ and $v_2(t)$ as described by the circuit diagram in Fig. 6.2.

Now having introduced the general device, let us proceed to a concrete model in the next section. The geometry was modelled and visualized with EM Studio by Computer Simulation Technology (CST). This software is based on the Finite Integration Technique (FIT) that is explained in Section 2.2. It utilizes conductor models that are similar to those derived in Section 4.1. The operators $C$ and $\tilde{C}$, the matrices of conductivities $M_{\sigma,\text{as}}$, permeability $M_\mu = M_\nu^{-1}$ and the coupling matrices $Q_{\text{sol}}, Q_{\text{str}}$ as well as the matrices of conductance $G_{\text{sol}}$ and resistance $R_{\text{str}}$ of the solid and stranded conductor were finally obtained from EM Studio using a second MATLAB software package named Symplegades. It has been developed at the department of Theorie Elektromagnetischer Felder (TEMF) at the University of Darmstadt.\footnote{The websites of CST Software and the TEMF institute are located at \url{http://www.cst.com} and \url{http://www.temf.de/}, respectively.}
6.1. TRANSFORMER

6.1.2 Modelling and Discretization

Geometry

Fig. 6.1(b) shows the geometry and Fig. 6.1(c) the dimensions of the single-phase transformer model. It is designed to meet a realistic example where the first coil consists of 358 and the second one of 206 copper strands wound around a laminated iron core of thickness 3 cm. Note that neither the strands nor the lamination of the core is individually modelled. The first is not necessary since each coil is implemented as a stranded conductor model and their windings are treated as a whole, see the explanation in Section 4.1.2. Their conductivity equals $5.77 \cdot 10^7 \text{ S/m}$. The lamination technique is represented by a solid conductor model that has different conductivities depending on the spatial directions; this yields the same effect. The conductivity in $x$- and $y$-directions of the core is $\sigma_x = \sigma_y = 5 \cdot 10^5 \text{ S/m}$, but vanishes in the $z$-direction $\sigma_z = 0$. The relative permeability equals $\mu_r = 1000$ (iron).

System of Equation

The matrices that were obtained by FIT have to be adequately assembled to yield the system (4.20); there are: the matrix of conductivities $M_{\sigma,\text{as}}$ and the curl-curl matrix $\tilde{C}M_{\nu,\nu}C$, both of dimension $N_{\text{fit}} \times N_{\text{fit}}$, where $N_{\text{fit}}$ denotes the number of edges in the grid. The coupling matrix $Q_{\text{str}}$ is of dimension $N_{\text{fit}} \times N_{\text{str}}$, where $N_{\text{str}} = 2$ equals the number of transformer coils. In addition we compute the dc-resistance matrix $R_{\text{str}}$, which is of dimension $N_{\text{str}} \times N_{\text{str}}$. We do not apply any regularization technique and thus we rely on the weak gauging property of the Krylov solver.

We already gave MATLAB spy-plots of the matrices $M_{\sigma,\text{as}}$ and $\tilde{C}M_{\nu,\nu}C$ in Fig. 2.6 in the section on the discretization of Maxwell’s equations. They were originally added to illustrate the band-structure, but this becomes now even clearer: all edges that are aligned in $z$-direction are not conductive because of the laminating
technique and thus the corresponding entries in the lower third of the diagonal of $M_{\sigma,\text{as}}$ equal zero and even more zeros arise from the fact that stranded conductors are present.

We combine the matrices to one system of equations, which is build according to the general mixed formulation (4.20) for solid and stranded conductors. In our case the iron core which is represented by the solid conductor is not connected to the circuit and thus the matrix $Q_{\text{sol}}$ vanishes and we obtain the reduced system (6.1) without a solid conductor coupling. The system reads

$$
\begin{pmatrix} \dot{C} M_{\nu} C & -Q_{\text{str}} \\ 0 & R_{\text{str}} \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{t}_{\text{str}} \end{pmatrix} + \begin{pmatrix} M_{\sigma,\text{as}} & 0 \\ Q_{\nu,\text{str}}^T & 0 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \bar{a} \\ \bar{t}_{\text{str}} \end{pmatrix} = \begin{pmatrix} 0 \\ u_{\text{str}} \end{pmatrix}
$$

(6.1)

and the less sparse formulation that was discussed in the Section 4.2.5 on sparsity is obtained by a block-Gaußian step

$$
\begin{pmatrix} \dot{C} M_{\nu} C & 0 \\ 0 & R_{\text{str}} \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{t}_{\text{str}} \end{pmatrix} + \begin{pmatrix} M_{\sigma,\text{fillin}} & 0 \\ Q_{\text{str}}^T & 0 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \bar{a} \\ \bar{t}_{\text{str}} \end{pmatrix} = \begin{pmatrix} G_{\text{str}} Q_{\text{str}} u_{\text{str}} \\ u_{\text{str}} \end{pmatrix},
$$

(6.2)

with $M_{\sigma,\text{fillin}} = M_{\sigma,\text{as}} + Q_{\text{str}} G_{\text{str}} Q_{\text{str}}^T$. In both cases the transformer is excited by a time-dependent voltage source $u_{\text{str}} = (v_1(t), v_2(t))^T$.

**Discretization Grid**

The discretization is performed with grids at three different degrees of fineness “coarse”, “medium” and “fine”. The coarse grid consists of $N = 864$ points...
6.1. TRANSFORMER

Grids | Points | Edges | Non-zero entries in the matrices
--- | --- | --- | ---
Coarse | 864 | 2592 | \( \tilde{C}M_\nu \) \( M_{\sigma,as} \) \( M_{\sigma,\text{fillin}} \) \( Q_{\text{str}} \) \( M_{\sigma,as}Q_{\text{str}} \)
Medium | 1680 | 5040 | 26856 | 240 | 56632 | 336 | 56
Fine | 15180 | 45540 | 53900 | 560 | 205240 | 640 | 120

Table 6.1: Discretization Grids. Properties of the discretization grid of the transformer model at three different degrees of fineness, all obtained by EM Studio by CST. Non-zero entries are obtained within MATLAB, \( M_{\sigma,\text{fillin}} = M_{\sigma,as} + Q_{\text{str}}G_{\text{str}}Q_{\text{str}}^T \).

The term \( M_{\sigma}Q_{\text{str}} \) in the last column has a special meaning. It allows us to decide whether a grid is sufficiently fine in terms of the grid condition \( G_2 \) or not. One can easily verify, that only the fine grid can fulfill \( G_2 \). In other words our analytic results regarding index and uniqueness would not apply to a MNA/FIT coupled model, if the coarse or medium grid are used. For the moment this is no problem, since we concentrate on the FIT model in this section.

6.1.3 Simulation Results

Parameters and Initial values

The current development of the transformer is examined on the time interval

\[
[0, \tau] = [0, 5 \cdot 10^{-2}],
\]

while excited by two alternating voltage sources that are applied to the primary and secondary coil. Both voltage sources operate basically at the same frequency

\[
f = 50\text{Hz},
\]

(6.3)

\[
\omega = 2\pi f,
\]

(6.4)
(a) Current density flowing in a coil
(b) Magnetic flux density in the core

Figure 6.4: Transformer Simulation. Current and flux densities in the transformer example, images are generated for illustration by EM Studio from CST.

but at different voltage levels
\[ v_1(t) = 380 \sin(\omega t) + 20 \sin(2\omega t), \]
\[ v_2(t) = 220 \sin(\omega t). \]
Additionally the voltage \( v_1 \) is perturbed by \( 20 \sin(2\omega t) \). However both vanish at the start time \( t = 0 \) and hence we may start the integration with the trivial initial value
\[ \tilde{a}_0 := 0. \]

Discussion of the Results

Simulations have been run for illustration purposes by EM Studio, see Fig. 6.4, but the actual results were obtained by three different MATLAB implementations. The functions \texttt{trafo\_sparse} and \texttt{trafo\_dense} rely on the \texttt{backeuler} algorithm. The first code integrates the sparse system (6.1) and the second the denser one (6.2). The sparse system (6.1) has been scaled before the actual computations have been made to achieve better convergence properties of the Krylov solver. On the other hand the denser system (6.2) offers a more efficient computation when the MQS curl-curl- and the coupling equations are successively computed; the current is obtained in a post-processing step. This has been implemented in \texttt{trafo\_post}. The headers of the implementations can be found in Appendix C.

We performed simulations for each grid using constant (Table 6.2) and variable step sizes (Table 6.3). The three implementations give numerically different approximations, but they differ less than the absolute tolerance (\( Atol = 10^{-3} \)) of
6.1. TRANSFORMER

<table>
<thead>
<tr>
<th>constant step sizes</th>
<th>time</th>
<th>step size</th>
<th>systems</th>
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<tr>
<td>coarse</td>
<td>13 s</td>
<td>10^{-3}</td>
<td>50</td>
</tr>
<tr>
<td>medium</td>
<td>41 s</td>
<td>10^{-3}</td>
<td>50</td>
</tr>
<tr>
<td>fine</td>
<td>13.3 m</td>
<td>10^{-3}</td>
<td>50</td>
</tr>
<tr>
<td>trafo_sparse</td>
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<td>10^{-4}</td>
<td>500</td>
</tr>
<tr>
<td>trafo_post</td>
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<td>10^{-4}</td>
<td>500</td>
</tr>
<tr>
<td>trafo_dense</td>
<td>2.0 m</td>
<td>10^{-4}</td>
<td>500</td>
</tr>
<tr>
<td></td>
<td>4.9 m</td>
<td>10^{-4}</td>
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</tr>
<tr>
<td></td>
<td>74 s</td>
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<td>97.8 m</td>
<td>10^{-4}</td>
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<td>10^{-4}</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7.8 h</td>
<td>10^{-4}</td>
<td></td>
</tr>
<tr>
<td></td>
<td>11.7 h</td>
<td>10^{-4}</td>
<td></td>
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</tbody>
</table>

Table 6.2: Computational Aspects of the Transformer Example (Constant Step Sizes). Computation time for each grid, time stepping parameters and number of solved linear systems by the Krylov solver; the bar “−” denotes that the Krylov solver did not converge.

the error control demands, see Section 5.3.1. On the other hand the Krylov solver does not always fulfill the relative tolerance (Rtol = 10^{-6}) although the norm of the relative residual is smaller than PCGtol = 10^{-8}. Thus the calculations with adaptive control would have needed fewer steps if PCGtol had been decreased. The scaling of the sparse system might change the effective condition number of the system and has to be chosen carefully, we scaled the first equation by a factor of 1/100.

Fig. 6.5(a) shows an exemplary plot of the current development of the problem on the medium grid that is obtained by trafo_dense using the clustered step size control, when the alternating voltages \( v_1(t), v_2(t) \) are applied as given in Fig. 6.5(b). We focus on the medium grid because its calculations still take a convenient time span while the fine grid needs significantly more computation time, see Table 6.2. In addition the PCG method did not reliably converge to a result within the required tolerance when using the fine grid (step size 10^{-3}). The absolute errors due to different grid sizes are illustrated in Fig. 6.5(e) and Fig. 6.5(f). The currents of the coarse grid differ up to maximum of 5A from the currents computed on the fine grid, while the medium grid’s currents still miss the results of the fine grid with a maximum of 2.7A. It depends on the application whether these errors are tolerable or not.

The conclusions of the efficiency of the implementations turns out to be very clear: the solving of the dense system is computational more expensive and hence trafo_dense and trafo_post are slower than trafo_sparse. This behaviour corresponds to the theoretical predictions of Section 4.2.5. The post-processing of the coupling equations in trafo_post yields faster results than the direct implementation of trafo_dense. The reason for this is the use of the PCG solver rather than the reduction of the linear system, since there are only two coupling equations and they are merely evaluations in the dense system. The advantage of trafo_post may increase, if there are more coupling equations.
CHAPTER 6. COMPUTATIONAL EXAMPLES

<table>
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<th>variable step size</th>
<th>coarse grid</th>
<th>medium grid</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>time</td>
<td>steps</td>
</tr>
<tr>
<td>standard control</td>
<td>15.5 m</td>
<td>1501 (155)</td>
</tr>
<tr>
<td>clustered control</td>
<td>13.2 m</td>
<td>1546 (102)</td>
</tr>
</tbody>
</table>

Table 6.3: Computational Aspects of the Transformer Example (Variable Step Sizes). Computation time of \texttt{trafo\_dense} for each grid, number of successful (rejected) steps and solved linear systems by the Krylov method (BICG)

The adaptive step size controls suggest step sizes in the order of $10^{-5}$ for the coarse and medium grid. The Fig. 6.5(c) and Fig. 6.5(d) show exemplary the developing of the step sizes of both controls integrating the dense system on the medium grid. The algorithm based on the clustered step size control needs according to Table 6.3 less time for more steps and is thus an adequate modification. This is reasonable since calculations of the system matrix (for the linear system involved) and the imposing of the boundary conditions are often avoided. The use of larger clusters might yield even better results for a particular problem but holds the risk of repeatedly underestimating the step size. The need for more steps becomes significant when the costs of the solution of the linear system increase. In this case the additional steps can become more expensive than the recalculation of the matrix pencil.

Conclusion

We have seen that all three implementations are generally feasible. The integration of the sparse system \texttt{trafo\_sparse} is obviously the most efficient implementation, but \texttt{trafo\_post} has also advantages. It integrates directly the curl-curl equation and calculates the currents in a post-processing step and this becomes beneficial for example, if a $\kappa$-regularization is applied. In such cases we can directly solve an ODE, while the other implementations still need to solve a DAE. Furthermore we are not dependent on the BICG method which is the reason why we chose this implementation in \texttt{OCATVE}. 
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(a) Currents computed with standard step size control, medium grid

(b) Excitation voltages \(v_1(t), v_2(t)\)

(c) Standard adaptive step size control, medium grid

(d) Clustered adaptive step size control, medium grid

(e) Abs. differences between the currents \(i_1, i_2\) of the coarse and fine grid, constant step size \(h = 10^{-4}\)

(f) Abs. differences between the currents \(i_1, i_2\) of the medium and fine grid, constant step size \(h = 10^{-4}\)

Figure 6.5: Plots of the Transformer Example. All figures are computed with the function \texttt{trafo\_dense}, parameters from Section 6.1.3.
6.2 RLC Oscillation Circuit

6.2.1 Introduction

The previous example was given to illustrate the field discretization by FIT and now the second example introduces the network approach by MNA. Later we will refine the circuit by a 3-D MQS FIT model of an inductor. For the moment the exemplary circuit contains a linear resistor, a linear inductor and a linear capacitor, hence it is referred to as a RLC circuit. The circuit is excited by a voltage source that operates at a constant voltage level.

In the next section the topology of the circuit will be described as shown in Fig. 6.6. The description utilizes the incidence matrices that were introduced in Section 3 and as a first result we shall formulate the MNA DAE-IVP.

6.2.2 MNA Circuit Model

The application of MNA to the circuit given in Fig. 6.6 requires knowledge on the elements and their interconnections. Those information are stored in the element relations and incidence matrices. Therefore we define the unknown state variables and consequently introduce four node potentials $e_1, e_2, e_3$ and $e_4$ that are augmented to a vector $e = (e_1, e_2, e_3, e_4)^T$ and two additional currents for the voltage source $j_V$ and the inductance $j_L$. Those unknowns are linked by the element relations and incidence matrices, that represent the topology of the circuit. They read

\[
A_C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C = C_1, \quad q(A_C e, t) = C \cdot A_C^T e, \quad (6.8a)
\]

\[
A_R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad G = \text{diag}(G_1, G_2), \quad r(A_R e, t) = G \cdot A_R^T e, \quad (6.8b)
\]

\[
A_V = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v(t) = u_{op}, \quad (6.8c)
\]

\[
A_L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = L_1, \quad \Phi(j_L, t) = L \cdot j_L, \quad (6.8d)
\]
6.2. RLC OSCILLATION CIRCUIT

with the conductances \( G_1 = R_1^{-1} \) and \( G_2 = R_2^{-1} \).

The example consists of linear elements only and thus we obtain a special case of the general MNA system (3.17). The resulting system is still a linear implicit differential-algebraic system and reads in matrix notation

\[
\begin{pmatrix}
A_C C A_C^T & 0 \\
0 & L_0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{d}{dt} e \\
-j_L n_V
\end{pmatrix} +
\begin{pmatrix}
A_R G A_R^T & A_L^T & A_V \\
-A_{L}^T & 0 & 0 \\
-A_{V}^T & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e \\
j_L \\
j_V
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
-v(t)
\end{pmatrix}.
\]

In this quite simple case we have only six equations and thus the system can be explicitly given. We obtain after inserting the incidence matrices

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C_1 & 0 & 0 & 0 \\
0 & 0 & 0 & L_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{d}{dt} e_1 \\
e_2 \\
e_3 \\
e_4 \\
j_L \\
j_V
\end{pmatrix} +
\begin{pmatrix}
G_1 & -G_1 & 0 & 0 & 0 & 1 \\
-G_1 & G_1 + G_2 & -G_2 & 0 & 0 & 0 \\
0 & -G_2 & G_2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
j_L \\
j_V
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
-u_{op}
\end{pmatrix}.
\]

One easily verifies that the second matrix is regular and hence the matrix pencil does trivially not vanish. Furthermore it is an (algebraic) index-1 system since the mass matrix is singular but diagonal. This observation is consistent with Theorem 3.1, since the circuits respects the topological conditions \( G_1 \) and \( G_2 \) (as well as \( H_1 \)). The general theorem was proved by the decomposition into differential and algebraic components. We shall derive this formulation here as well because it allows us to deduce consistent initial values.
Semi Explicit Formulation

The semi explicit MNA form (6.20) was obtained with a projector $Q_C$ onto the kernel of $A_C$ and its complementary projection $P_C$. The corresponding matrix are easy to determine in the simple case of our RLC circuit, they read

$$Q_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ and $$P_C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Applying $P_C$ to equation (6.10) yields the differential part of the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & L_1 \end{pmatrix} \dot{y}_{mna} + \begin{pmatrix} G_1 & -G_1 & 0 & 0 \\ -G_1 & G_1 + G_2 & -G_2 & 0 \\ 0 & -G_2 & G_2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} y_{mna} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

(6.11a)

with variables $y_{mna} = (PCe, i_L)$ and $z_{mna} = (QCe, jV)$. The algebraic part is obtained with the help of $Q_C$

$$\begin{pmatrix} G_1 & -G_1 & 0 & 0 & 1 \\ -G_1 & G_1 + G_2 & -G_2 & 0 & 0 \\ 0 & -G_2 & G_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} z_{mna}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} y_{mna} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -u_{op} \end{pmatrix}.$$ 

(6.11b)

The regularization of the first matrix allows us to solve the equation (6.11b) for the algebraic variable $z_{mna}$ and this gives

$$z_{mna} = (u_{op}, -R_1 jL + u_{op}, -R_1 R_2 (G_1 + G_2) jL + u_{op}, 0, -jL)^T.$$  

(6.12)
Now the algebraic variable $z_{\text{mna}}$ is uniquely determined in dependence of the differential components. This is only $jL$ in this particular case, since the node potential $e_4$ is not regarded in equation \((6.11b)\). Finally the algebraic components of the initial value $x_{\text{mna},0}$ are fixed, if the two components $e_{4,0}$ and $jL_0$ are supplied. We will exploit this later to construct a consistent initial value.

The MNA DAE-IVP is now completely described except for the device parameters. They have to be chosen in correspondence with the FIT refinement and thus are also given later in Section \([6.2.4]\).

### 6.2.3 FIT/MNA Coupled Model

The FIT/MNA coupled example shall illustrate the refinement of an inductance. We replace the simple inductor model that was considered in the MNA by a FIT model. The stranded conductor model is the reasonable choice. As explained before the stranded conductor corresponds to an inductance with an additional resistance in series, see Fig. \([4.5(b)]\). Therefore we also remove the resistor $R_2$ and consequently the unnecessary node potential $e_3$ from the original circuit since the FIT model is inserted between $e_2$ and $e_4$. This change requires a new circuit description with only three node potentials $e = (e_1, e_2, e_4)^T$ that is shown in Fig. \([6.7]\).

We shall present all numeric approaches that were discussed in this treatise. The monolithic, dynamic iteration and multi-rate co-simulation will be utilized in the following to simulate the same coupled model.

#### Monolithic Approach

The monolithic coupling approach as introduced in Section \([5.2.2]\) combines both the FIT and the MNA problems in one system of equations. The general form
is described by system \(5.3\). In this oscillation circuit example the system is much simpler since the elements are linear and there are neither inductors left nor voltage sources present in the circuit and no coupling is established to solid conductors.

The incidence matrices are similar to the ones in plain MNA, with the only difference that the third line that corresponds to the node potential \(e_3\) is removed from all matrices. The particular incidence matrix \(A_L\) of the inductor is replaced by

\[
A_{\text{str}} := \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}
\]

(6.13)

for the stranded conductor coupling. These definitions yield the following coupled system, written here once more in matrix notation

\[
\begin{pmatrix}
A_C A_T^L & 0 & 0 & 0 \\
0 & 0 & 0 & Q^T_{\text{str}} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{\text{fit,as}}
\end{pmatrix}
\begin{pmatrix}
ed \\
\dot{\nu}_{\text{str}} \\
\dot{\nu} \\
\dot{\alpha}
\end{pmatrix}
+ \begin{pmatrix}
A_R G A_T^R & A_{\nu} & 0 \\
-A_{\nu}^T R_{\text{str}} & 0 & 0 \\
-A_{\nu}^T & 0 & 0 & 0 \\
0 & -Q_{\text{str}} & 0 & K_{\nu}
\end{pmatrix}
\begin{pmatrix}
ed \\
\nu_{\text{str}} \\
\nu \\
\dot{\alpha}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
-v(t) \\
0
\end{pmatrix}
\]

(6.14)

The upper left blocks of both matrices are basically the ones from MNA, but they are extended by \(N_{\text{fit}}\) field equations and unknowns. These additions are the significant larger part compared with the contribution from circuit analysis, although they are represented by only one row and column.

System (6.14) is constructed in the m-file rlc_monono and is finally integrated by the backeuler algorithm, see Appendix C.

Dynamic Iteration

The adequate notation for the dynamic iteration has been introduced in section 5.2.3. It utilizes two separate systems for each sub-problem. The first sub-problem is the MNA IVP and its system is of the type \(5.13a\).

In this particular example of the oscillation circuit the inductor is replaced by an additional current source \(\lambda = \nu_{\text{str}}\) with the corresponding incidence matrix \(A_{\lambda} = A_{\text{str}}\). This coupling has only minor impact on the MNA system, it is constructed in the same manner as before and the result reads

\[
\begin{pmatrix}
A_C A_T^L & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
ed \\
\dot{\nu}
\end{pmatrix} + \begin{pmatrix}
A_R G A_T^R & A_{\nu} \\
-A_{\nu}^T & 0
\end{pmatrix}
\begin{pmatrix}
ed \\
\nu
\end{pmatrix} = \begin{pmatrix}
-A_{\text{str}} \nu_{\text{str}} \\
-v(t)
\end{pmatrix}
\]

(6.15)
with only one resistance \( R_1 = G_1^{-1} \), since the other one is implicitly included in the FIT model.

The description of the field system is analogue to the one given for the transformer, see equation (6.1). The differences in the notation are marginal: \( Q_{\text{str}} \) is now a column vector, since we couple only to one coil and the excitation current does not originate from a directly attached voltages source, but from the circuit description above.

\[
\begin{pmatrix}
\hat{C} M_{\nu} C & -Q_{\text{str}} \\
0 & R_{\text{str}}
\end{pmatrix}
\begin{pmatrix}
\vec{a} \\
\vec{t}_{\text{str}}
\end{pmatrix}
+
\begin{pmatrix}
M_{\sigma,\text{as}} & 0 \\
Q_{\text{str}}^T & 0
\end{pmatrix}
\frac{d}{dt}
\begin{pmatrix}
\vec{a} \\
\vec{t}_{\text{str}}
\end{pmatrix}
=
\begin{pmatrix}
0 \\
A_{\text{str}}^T e
\end{pmatrix}.
\] (6.16)

The MATLAB file rlc_cosim constructs the example and finally calls the function gaussseidel for the integration.

### 6.2.4 Simulation Results

**Parameters**

We have to choose the parameters of both, the MNA and FIT/MNA coupled examples carefully and in accordance to each other since we want to compare the simulation results. The inductance \( L_{\text{str}} \) of the FIT example can be calculated from the curl-curl- and the coupling matrix according to equation (4.35). The additional stranded conductor resistance is given by \( R_{\text{str}} \) and thus the following two definitions give comparable simulations

\[
L_1 := L_{\text{str}} = 0.9369 \text{ H},
\]

\[
R_2 := R_{\text{str}} = 1 \Omega.
\] (6.17a)

(6.17b)

All field quantities are deduced from the same FIT model as used in the transformer example utilizing the medium grid, although it is analytical not sufficiently fine. The reason is that the computational costs of the medium grid are significantly lower and the introduction of the fineness was merely a technical assumption.

We are not interested in the whole transformer anymore but in an inductor, so we couple to the first coil only. Consequently the second one is disregarded and the model corresponds to an inductor with an iron core. This will yield a slightly different behaviour than the equivalent circuit with the inductance \( L_{\text{str}} \) because the effects within the iron core are also regarded by the field model. We can interpret this fact as an additional benefit of the refined modelling, since this behaviour has no representation in the plain MNA circuit.


### Table 6.4: Computational Aspects of the RLC Example (Variable Step Sizes).

<table>
<thead>
<tr>
<th>variable step sizes</th>
<th>time</th>
<th>steps</th>
<th>step size</th>
<th>systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>rlc_mna</td>
<td>1 s</td>
<td>81 (7)</td>
<td>6.3·10⁻⁷ to 2.0·10⁻³</td>
<td>176</td>
</tr>
<tr>
<td>rlc_mono</td>
<td>3.4 h</td>
<td>1468 (0)</td>
<td>10⁻⁵ to 1.8·10⁻⁴</td>
<td>4404</td>
</tr>
</tbody>
</table>

The other circuit parameters for the example were chosen as

\[
C_1 := 1 \cdot 10^{-4} \text{ F,} \tag{6.17c}
\]
\[
R_1 := 10 \text{ \Omega,} \tag{6.17d}
\]
\[
v(t) := 250 \text{ V.} \tag{6.17e}
\]

**Initial Values**

The RLC oscillation circuit is simulated within the time interval

\[ [0, \tau] = [0, 0.1], \]

and its initial values are consistently predefined in accordance to equation (6.12) and read

\[
e_{1,0} := 250 \text{ V,} \tag{6.18a}
\]
\[
e_{2,0} := 250 \text{ V,} \tag{6.18b}
\]
\[
e_{3,0} := 250 \text{ V,} \tag{6.18c}
\]
\[
e_{4,0} := 0 \text{ V,} \tag{6.18d}
\]
\[
j_{L,0} := 0 \text{ A,} \tag{6.18e}
\]
\[
j_{V,0} := 0 \text{ A.} \tag{6.18f}
\]

They are used for both the plain MNA and the FIT/MNA coupled problem, where applicable. The initial current \(j_{L,0}\) is set to zero to simplify the choice of the initial field value. We may now start with the trivial initial MVP

\[ \vec{a}_0 := 0. \tag{6.18g} \]

**Discussion of the Results**

Two fundamental different models of the same RLC circuit have been presented: the first was obtained by a plain MNA description and the second by a refined approach, that couples a FIT and MNA model. The MNA system has been constructed by `rlc_mna` and integrated by `ode15s`, while the (monolithic) coupled
6.2. RLC OSCILLATION CIRCUIT

<table>
<thead>
<tr>
<th></th>
<th>time</th>
<th>steps</th>
<th>step size $h$</th>
<th>systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>rlc_mono</td>
<td>24.5 m</td>
<td>100</td>
<td>$10^{-3}$</td>
<td>100</td>
</tr>
<tr>
<td>rlc_mono</td>
<td>88.5 m</td>
<td>1000</td>
<td>$10^{-4}$</td>
<td>1000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>time</th>
<th>steps</th>
<th>macro size $H$</th>
<th>systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>rlc_cosim ($l = 0$)</td>
<td>2.1 m</td>
<td>100 / 100</td>
<td>$10^{-3}$</td>
<td>100</td>
</tr>
<tr>
<td>rlc_cosim ($l = 0$)</td>
<td>17.1 m</td>
<td>1000 / 1000</td>
<td>$10^{-4}$</td>
<td>1000</td>
</tr>
<tr>
<td>rlc_cosim ($l = \text{auto}$)</td>
<td>6.0 m</td>
<td>100 / 100</td>
<td>$10^{-3}$</td>
<td>397</td>
</tr>
<tr>
<td>rlc_cosim ($l = \text{auto}$)</td>
<td>28.3 m</td>
<td>1000 / 1000</td>
<td>$10^{-4}$</td>
<td>3002</td>
</tr>
<tr>
<td>rlc_cosim ($l = 10$)</td>
<td>7.1 m</td>
<td>100 / 100</td>
<td>$10^{-3}$</td>
<td>1100</td>
</tr>
<tr>
<td>rlc_cosim ($l = 10$)</td>
<td>36.7 m</td>
<td>1000 / 1000</td>
<td>$10^{-4}$</td>
<td>11000</td>
</tr>
</tbody>
</table>

Table 6.5: Computational Aspects of the RLC Example (Constant Step Sizes). Computation time, number of (macro/micro) steps, step sizes and solved linear systems by the Krylov solver

The plain circuit model was merely given to underline the plausibility of the other (coupled) results. We will focus in the following on the different approaches (monolithic, dynamic iteration and multi-rate co-simulation) of the coupled model only.
We performed simulations of each approach at two different constant step sizes, $10^{-3}$ and $10^{-4}$. The monolithic problem is integrated with the same code as before (rlc\_mono), but now invoking backeuler with the parameter hcontrol = 0. We compare the different approaches with the adaptively computed “reference” solution. This is reasonable since the constant step sizes are larger than the minimal step size of the adaptive one. The absolute errors are given in Fig. 6.9.

According to Table 6.3 the monolithic simulation turns out to be the computational most expensive approach, but it scales very well: a decrease of the step size by one order of magnitude yields an error that decreases by the same factor while the calculation time increases slightly less.

The co-simulations have been performed by rlc\_cosim that utilizes the function gaussseidel. In general we distinguish here between (macro) step size $H$ and micro step sizes $h$ in the co-simulation to allow multi-rate integration, but this is unnecessary in this particular example. The circuit does not demand smaller step-sizes than the field, see Table 6.5. The adaptive integrator (ode15s) which is called within gaussseidel needs only one micro step to fulfill the tolerances. This confirms the prior observation that this example has no multi-rate potential. Although this conclusion is not generalizable, since the significance of this example circuit is limited, we cannot generally suspect the magnetoquasistatic field and electric circuit to have different time constants. Therefore we find dividing the circuit into parts of different dynamics a more promising approach. Each of those parts may still include field models.

Table 6.5 shows that the co-simulation approaches are clearly advantageous regarding the computational costs when compared to the monolithic simulation at the same step size, while the errors are of the same - admittedly large - order as Fig. 6.9 shows. The co-simulations benefit from the simpler structure of the linear systems that have to be solved by the Krylov solver. But the low order of our integration scheme and the lack of multi-rate prevent better results at the given step sizes. It is noticeable that the iterated result is inferior than the one without iterating, when compared to the monolithic reference solution. This discrepancy decreases for smaller step sizes. The multi-rate co-simulation yields even for the largest step size comparatively good approximations, but profits less from step size decreases. The dynamic iteration approach converges at most after 3 iterations to its final approximation and more iterations (e.g. $l = 10$) do not noteworthy change it.

\footnote{\textsuperscript{2}The term “multi-rate co-simulation” becomes misleading here since micro and macro step sizes are equal.}
6.2. RLC OSCILLATION CIRCUIT

Figure 6.8: Plots of the RLC Example (Variable Step Sizes). Integrations of the MNA and monolithic coupled problem using ode15s and backeuler (clustered) on the medium grid with the the parameters from Section 6.2.4.
CHAPTER 6. COMPUTATIONAL EXAMPLES

Figure 6.9: Absolute Errors of the RLC Example. The plots show the errors of each node potential with respect to the reference solution; parameters as given in Section 6.2.4.
6.3 Summary

In this chapter we have demonstrated that the FIT/MNA coupling as proposed in Chapter 5 is at least capable of modelling configurations that are equal to those from plain circuit analysis, but the presented methods are more general and even feasible for more complex problems. We will give an outlook on how this situation can be further improved in the next chapter.
Chapter 7

Summary and Outlook

7.1 Summary

We presented the general theory of electromagnetic field and circuit simulations and introduced the corresponding FIT and MNA approaches. Both problems were formulated in detail and their individual properties were analyzed. It has been shown that both problems are index-1 DAEs provided that several simple conditions are fulfilled. Furthermore a voltage-driven reformulation of the field problem has been derived that is analytically equivalent to an ODE.

A consistent coupling formulation of the FIT/MNA system has been proposed and two approaches for its numerical treatment were discussed. The monolithic and co-simulation approaches have been found theoretically and practically feasible. Both were tested using an example circuit refined by a 3-D conductor model, that was integrated by the supplied demonstration code “FIDES”. Furthermore the co-simulation software has been successfully ported to OCTAVE and adapted to fit into the COMSON demonstrator platform\footnote{COMSON is an European research training network on “Coupled Multiscale Simulation and Optimization in Nanoelectronics”, see \url{http://www.comson.org}}.

7.2 Outlook

This treatise achieved a profound bases of the FIT/MNA coupling but several interesting question were raised, but not completely answered. The existence of an ODE that is equivalent to the MQS curl-curl equation for solid and stranded conductors was proved non-constructively using the Kronecker’s Theorem. A constructive procedure would be of great value for the theoretical analysis and possibly even beneficial for the numerical treatment.
On the one hand the demonstration code still uses the BICG method but Algebraic Multi-Grid solver are very successfully used for FIT problems and can easily be attached, see for example [12]. Furthermore the code bases on the implicit Euler method and should be enhanced by a higher order method. This will improve the convergence properties of the co-simulation and hence give better results. On the other hand the software package Symplegades comes already with an optimized SDIRK method, so a simulator coupling of this package to the COMSON demonstrator platform is a reasonable alternative.

The computational examples were only given to prove the coupling concept and were therefore very simple. More complex examples should be studied, that means larger and for example non-linear problems. Furthermore the parameters of the circuit elements should be chosen from micro- rather than macro-electronics like those in this treatise, since the MNA makes especially sense in those scenarios. One could start with the circuit given in Fig. 7.1. it shows an AC-DC converter that is an example of a more sophisticated field/circuit problem.

Finally the potential of the multi-rate and co-simulation approaches should be discussed in more detail. The current results of this treatise show that the magnetoquasistatic field and electric circuit do not have different time constants in general and instead a partitioning of the circuit seems more promising. More numerical tests will give further insight.
Appendix A

Linear Algebra

We need several simple definitions and deductions from linear algebra in this treatise. We collect those results in this chapter. We denote the transpose of a vector $x$ by $x^T$ and analogously for a matrix $A$ by $A^T$. The corresponding entries of vectors and matrices are always real numbers ($\mathbb{R}$).

A.1 Pseudoinverse

The FIT discretization yields singular matrices that consequently do not have an inverse, although we are in the need of it. Since in our case a subset of the properties of an inverse matrix is sufficient, we define a matrix called pseudoinverse that has the needed properties.

Definition A.1 (Moore-Penrose Pseudoinverse). Let $A$ denote a not necessarily square matrix. Then $A^+$ denotes the uniquely defined Moore-Penrose pseudoinverse, if it satisfies the following conditions

\[
AA^+ A = A, \\
A^+ AA^+ = A^+, \\
(AA^+)^T = AA^+, \\
(A^+ A)^T = A^+ A.
\]

Theorem A.1. Let $K = U\Sigma V^T$ denote the singular value decomposition of $A$, where $\Sigma$ is a diagonal matrix, then it holds that $A^+ = U\Sigma^+ V^T$.

The pseudoinverse $\Sigma^+$ of a diagonal matrix $\Sigma$ can be calculated by

\[
(\Sigma^+)^{ij} := \begin{cases} 
1/(\Sigma)^{ij} & \text{if } i = j \text{ and } (\Sigma)^{ij} \neq 0, \\
0 & \text{else}. 
\end{cases} \tag{A.1}
\]
A.2 Definiteness and Symmetry

The FIT matrices belong to special classes of matrices and thus have several properties that are essential for our analysis. In the following $A$ and $A_1, A_2, \ldots$ always denote square matrices whereby the matrix $B$ is not necessarily square.

A.2.1 Definitions

Let us give in the following the well-known definitions of a positive (semi-)definite and symmetric matrix, respectively.

**Definition A.2.** A square matrix $A$ is called **positive definite** iff for all non-zero vectors $x \neq 0$ it holds that $x^T A x > 0$.

**Definition A.3.** A square matrix $A$ is called **positive semi-definite** iff for all non-zero vectors $x \neq 0$ it holds that $x^T A x \geq 0$.

**Definition A.4.** A square matrix $A$ is called **symmetric** iff $A = A^T$.

A.2.2 Simple Propositions

From the definition above several simple results can be elementarily derived. We state those results in the following without a proof.

**Proposition A.1.** A positive definite matrix $A$ is invertible and its inverse denoted by $B^{-1}$ is also positive definite.

**Proposition A.2.** If a matrix $A$ has independent columns then the matrix product $A^T A$ is positive definite, otherwise it is only positive semi-definite.

**Proposition A.3.** If two matrices $A_1$ and $A_2$ are symmetric positive definite then so is the sum $A_1 + A_2$.

**Corollary A.1.** If two matrices $A_1$ and $A_2$ are symmetric positive semi-definite then so is the sum $A_1 + A_2$. If only one matrix, let us say $A_1$ is strictly positive definite then this is also true for the sum.

**Proposition A.4.** Let $B$ denote a rectangular matrix with independent columns and $A$ a symmetric positive definite matrix, then the product $B^T A B$ is symmetric positive definite.

**Proposition A.5.** Let $B$ denote a rectangular matrix and $A$ a symmetric positive definite matrix, then it holds that $\ker B^T A B = \ker B$. 
Appendix B

Differential Equations

B.1 Ordinary Differential Equations

This treatise concerns numerical solutions of differential problems. Those problems are in the simplest case ordinary differential equations (ODEs)

\[
\dot{y}_i = f_i(t, y_1, \ldots, y_n), \quad i = 1, \ldots, n
\]  

(B.1)

which describe the unknown function \( y_i : [0, \tau] \to \mathbb{R} \) via its derivative with respect to time \( \dot{y}(t) = \frac{d}{dt}y(t) \) for a given initial value

\[
y_i(0) = y_{i0} \in \mathbb{R}, \quad i = 1, \ldots, n.
\]  

(B.2)

We call the system (B.1) with a corresponding initial values (B.2) an ordinary differential equation initial value problem, for short ODE-IVP. It reads in vector notation

\[
\dot{y}(t) = f(t, y(t)), \quad y(0) = y_0 \in \mathbb{R}^n,
\]  

(B.3)

where \((t, y) \in D\) and \(f : D \to \mathbb{R}^n\) with domain \(D = [0, \tau] \times \mathbb{R}^n\). Note that \(y\) shall always denote a differential variable in this treatise.

The simplest case of an ODE-IVP is the linear initial value problem, which reads in vector notation

\[
\dot{y}(t) = K \cdot y(t) + k(t), \quad y(0) = y_0,
\]  

(B.4)

with a constant coefficient matrix \(K \in \mathbb{R}^{n \times n}\) and time-dependent function \(k(t) : [0, \tau] \to \mathbb{R}^n\).

We will quote in the following the important from from Picard-Lindelöf on the uniqueness of the ODE-IVP without a proof. See Walter [58] for the proof and [24] for more information on the numerical analysis of ODEs.
Definition B.1 (Lipschitz condition). Let us consider system (B.3). The right-hand side function $f$ is said to fulfills the Lipschitz condition on the domain $D$, if the following assessment holds

$$||f(t,y^1) - f(t,y^2)|| \leq L \cdot ||y^1 - y^2||, \quad \forall (t,y^1), (t,y^2) \in D. \quad (B.5)$$

Theorem B.1 (Picard-Lindelöf, 1894). If the right-hand side function $f$ is continuous on the domain $D$ and it fulfills the Lipschitz condition, then the ODE-IVP (B.3) has a unique solution. This solution is continuously differentiable on the whole interval $[0,\tau]$.

B.2 Differential-Algebraic Equations

The differential problems in this treatise do not yield explicit systems of the introduced type (B.3), but implicit systems of type

$$F(t, \dot{x}(t), x(t)) = 0, \quad (B.6)$$

where $F : [0,\tau] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denotes a general non-linear mapping and $x(t) : [0,\tau] \to \mathbb{R}^n$ the time-dependent unknown function. In our case the discretized field problem and the linear network analysis yield linear systems of implicit differential equations

$$M \cdot \dot{x}(t) + K \cdot x(t) = k(t), \quad x(0) = x_0, \quad (B.7)$$

where $M, K \in \mathbb{R}^{n \times n}$ denote constant coefficient matrices. The first is matrix is called mass matrix and the latter is sometimes referred to as the stiffness matrix. Unlike before the initial value has to be choose consistently to yield a well-posed problem [42]. The time-dependent unknown $x(t)$ is described by its time derivative $\dot{x}(t)$ and the function $k(t) : [0,\tau] \to \mathbb{R}^n$.

If the mass matrix $M$ is regular, system (B.7) represents a linear implicit system of ODEs and can be transformed into the explicit ODE system

$$\dot{x}(t) = M^{-1} (-K \cdot x(t) + k(t)), \quad x(t_0) = x_0. \quad (B.8)$$

On the other hand, if $M$ is singular, equation (B.7) represents a system of differential-algebraic equations (DAE). Note that $x$ shall denote in this treatise a variable with differential ($y$) and algebraic parts ($z$). The structural properties are analyzed in the following by investigating the matrix pencil and the singular mass matrix, which yields the DAE index concept.
APPENDIX B. DIFFERENTIAL EQUATIONS

B.2.1 Matrix Pencil

Definition B.2. [Matrix Pencil] A pair of matrices $M, K$ is said to form a regular matrix pencil

$$[M, K] := K + \alpha M, \quad (B.9)$$

if there is a parameter $\alpha \in \mathbb{R}$, such that $\det(K + \alpha M) \neq 0$.

Definition B.3 (Characteristic Polynomial). Let $M, K \in \mathbb{R}^{n \times n}$ denotes a pair of matrices, then the polynomial $p$ of degree $n$, defined by

$$p(\alpha) := \det(K + \alpha M), \quad (B.10)$$

is referred to as the characteristic polynomial of $[M, K]$.

Proposition B.1. Let the matrices $M, K \in \mathbb{R}^{n \times n}$ form a regular matrix pencil $[M, K]$, then there is an $\alpha_1 > 0$ such that $K + \alpha_1 M$ is regular.

Proof. Since the matrix pencil is regular, there is an $\alpha_0$ such that it holds $\det(K + \alpha_0 M) \neq 0$. The calculation of the determinant yields the characteristic polynomial $p(\alpha)$ of finite degree $n$. This polynomial $p(\alpha)$ does not equal the zero polynomial, because $p(\alpha_0) \neq 0$. Therefore $p(\alpha)$ has maximally $n$ roots in $\mathbb{R}$ and thus there is an $\alpha_1 > 0$ such that $p(\alpha_1) \neq 0$. \hfill $\square$

If the pencil is not regular or equivalently $\det(K + \alpha M)$ vanishes identically, it is not even reasonable to try to solve these systems numerically in general and most ODE methods reject these problems automatically, since it is impossible to find a timestep $\Delta t$ so that $M/\Delta t + K$ is regular. On the other hand, if the pencil is regular, the system is called solvable and the DAE index concept is well defined, Gear and Petzold [30].

B.2.2 DAE Index

Since differential/algebraic equations are not ODEs, as provocatively formulated in Petzold [47], there is no explicit form. Alternatively one is interested in a form in which the algebraic and differential parts are separated from each other. The idea is to find a transformation with regular matrices $P_x$ and $Q_x$

$$P_x M \begin{array}{c} x \end{array} Q_x^{-1} \begin{array}{c} \dot{x} \end{array} + P_x K \begin{array}{c} x \end{array} Q_x^{-1} \begin{array}{c} x \end{array} = P_x k(t), \quad (B.11)$$

which reduce the matrix pencil to a normal form.
Theorem B.2 (Kronecker, 1890). Let \([M, K]\) form a regular matrix pencil, i.e. the system of type (B.7) is uniquely solvable. Then there exist regular matrices \(P_x\) and \(Q_x\) so that the Jordan decomposition holds
\[
M^* = P_x M Q_x = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}, \quad K^* = P_x K Q_x = \begin{pmatrix} W & 0 \\ 0 & I \end{pmatrix},
\]
where \(I\) denotes the unit matrix, \(W\) is a regular square matrix and finally \(J = \text{diag}(J_1, \ldots, J_k)\) is block-matrix and consists of \(k\) Jordan blocks \(J_i\) to the eigenvalue 0.

The Jordan blocks \(J_i\) are of the type
\[
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
\vdots & \ddots \\
0 & 1 & 0
\end{pmatrix},
\]
and thus matrix \(J\) is nilpotent.

Definition B.4 (Index of Nilpotency). If \(\nu\) denote the smallest number such that \(J^\nu = 0\), but \(J^{\nu-1} \neq 0\), then \(\nu\) is called \(J\)'s index of nilpotency.

We decompose the transformation matrices further to directly address the blocks that correspond to differential and algebraic parts
\[
k^* = P_x k = \begin{pmatrix} I \\ 0 \end{pmatrix} (I \underleftarrow{P_x} 0) P_x k + \begin{pmatrix} 0 \\ I \end{pmatrix} (0 \underleftarrow{P_x} I) P_x k,
\]
and
\[
x = Q_x x^* = Q_x \begin{pmatrix} y \\ z \end{pmatrix} = Q_x \begin{pmatrix} I \\ 0 \end{pmatrix} y + Q_x \begin{pmatrix} 0 \\ I \end{pmatrix} z.
\]

This allows us to rewrite the nilpotent Jordan matrix \(J\) and the regular \(W\) in terms of the new matrices from equation (B.14) and yields
\[
I = P_y M Q_y,
\]
\[
J = P_z M Q_z,
\]
\[
W = P_y K Q_y,
\]
\[
I = P_z K Q_z.
\]
Finally we obtain a decoupled formulation for systems of nilpotency \( \nu \), when the decompositions from above are used

\[
\begin{align*}
\dot{y}(t) + Wy(t) &= P_y k(t), \\
J \dot{z}(t) + z(t) &= P_z k(t).
\end{align*}
\] (B.19a, B.19b)

This differential-algebraic system is often referred to as the **Kronecker normal form** (KNF) of system \((B.7)\). Let us concentrate on the simpler special case, where the nilpotency \( \nu \) equals 1, since this holds for the systems that are in focus of this treatise. In this case the Jordan block \( J \) vanishes and the semi-explicit KNF reads

\[
\begin{align*}
y(t) + Wy(t) &= P_y k(t), \\
z(t) &= P_z k(t).
\end{align*}
\] (B.20a, B.20b)

These systems of nilpotency \( \nu = 1 \) behave almost like ODEs. That means they can be solved with implicit integrators, but systems of higher index become increasingly difficult to solve. Since the numerical behaviour of a DAE system depends so much on \( \nu \), the following definition is made:

**Definition B.5 (Algebraic Index).** Let \([M, K]\) describe the regular matrix pencil of the linear DAE \((B.7)\). If \( M \) is regular, i.e., in the ODE case, then the algebraic index is 0 or else it is equal to the nilpotency \( \nu \) of the matrix \( J \) in \((B.19)\).

There are various different index concepts, for example the perturbation index is important from a numerical point of view. Although we will not cover them all, we want to introduce one more concept: the **differential index**. It is closely related to the algebraic but and its application to some example is advantageous.

**Definition B.6 (Differential Index).** Let us consider the abstract system \((B.6)\).

The differential index of this system is the smallest number \( \nu \), such that the following time derivatives up to order \( \nu \)

\[
F(t, \dot{x}, x) = 0
\] (B.21)

\[
\frac{d}{dt}F(t, \dot{x}, x) = 0
\] (B.22)

\[
\vdots
\]

\[
\frac{d^\nu}{dt^\nu}F(t, \dot{x}, x) = 0
\] (B.24)

uniquely determines \( \dot{x} \) as a continuous function in \( x \) and \( t \). This is called an underlying ODE.

We focus on the algebraic index in our analysis. This is no limitation, since all these indices are equal in the present case of a DAE with linear constant coefficients. This is shown in \([32]\) for the special case of the modified nodal analysis, but could be generalized with analogous arguments.
Appendix C

Matlab Headers

Listing C.1: *Import FIT*. MATLAB m-file that imports a FIT problem from several mat-files

```matlab
function fitmodel=fit_import(path)
% FIT_IMPORT Import a FIT model from mat-files, returning a structure
%
% FITMODEL = FIT_IMPORT(path)
% Input parameters:
% PATH : path to the model
% Output parameters:
% FITMODEL : structure containing the matrices
```

Listing C.2: *Dirichlet Boundary Conditions*. MATLAB m-file that applies the Dirichlet Boundary conditions to the FIT matrices

```matlab
function S=fit_dirichlet(K, idx)
% FIT_DIRICHLET Applies the Dirichlet boundary conditions
to the (not necessarily sparse) FIT square matrix K, using the
indexes from the vector idx and returns a sparse matrix S.

S=FIT_DIRICHLET(K,IDX)
Input parameters
K : Old matrix pencil
IDX : Vector of indices
Output parameters
S : New matrix pencil incl. Dirichlet conditions
```
Listing C.3: *Import MNA.* MATLAB m-file that imports a MNA problem from a single mat-file

```matlab
function mnamodel=mna_import(file)
% MNA_IMPORT Import a MNA model from file, returning a structure
% MNAMODEL = MNA_IMPORT(FILE)

% Input parameters:
% FILE : a filename
% Output parameters:
% MNAMODEL : a structure
% Set a global variable 'verbose' to a file id ('fid')
% to see or save debug information
```

Listing C.4: *BICG method.* MATLAB m-file for use in conjunction with Backuler. It computes the actual backward Euler integration step with the BICG method

```matlab
function [y,flag,niter]=runbicg(Ash,B,Zsh,shrink,replace,steps,pcgtol,...
  fvalues,yold,yguesses)
% RUNBICG Compute one or more integration steps for backeuler
% using a Krylov solver.

% [Y,FLAG,NITER] = RUNBICG(ASH,B,ZSH,SHRINK,REPLACE,STEPS,PCGTOL,...
% FVALUES,YOLD,YGUESSES)

% Input parameters:
% ASH : shrinked matrix pencil with BC applied
% B : mass matrix divided by time step
% ZSH : shrinked preconditioner
% SHRINK : indeces of non zero rows/columns
% REPLACE : indeces of rows/columns for the BC
% STEPS : number of integration steps
% PCGTOL : tolerance for the solver
% FVALUES : function evaluations of f at each time step
% YOLD : vector of the old solution Y
% YGUESSES : initial guesses for pcg (optional)

% Output parameters:
% Y : approximate solutions at every time step
% FLAG : = 0: solver converged to the desired tolerance
%        > 0: solver ended erroneous
% NITER : Number of iterations

% Max. iterations of bicg are hardcoded to maxit=5000.
% Set a global variable 'verbose' to a file id ('fid')
% to see or save debug information
```
Listing C.5: Backward Euler. Matlab m-file implementing the backward Euler method with and without step size control

```matlab
function [t,y,istat,idid] = backeuler(M,K,Q,shrink,...
    replace,interest,odefun,...
    tspan,y0,rtol,atol,h0,hcontrol)

% BACKEULER Solves implicit FIT/MNA coupled DAEs of type
% M*Y'(T)+K*Y(T)=Q*F(T) with the adaptive backward Euler.
% 
% [T,Y,ISTAT,IDID] = BACKEULER(M,K,Q,SHRINK,...
% REPLACE,INTEREST,ODEFUN,...
% TSPAN,Y0,RTOL,ATOL,H0,CONST)
% 
% Input parameters:
% M : constant matrix M
% K : constant matrix K
% Q : constant matrix Q
% SHRINK : indeces of non zero rows/columns
% REPLACE : indeces of rows/columns for the BC
% INTEREST : indeces of variables of interest
% in addition to those of FIT
% ODEFUN : name of right hand side F
% TSPAN : vector [T0,TEND]
% Y0 : initial value (dimension n*1)
% RTOL,ATOL : relative and absolute tolerance
% H0 : initial step size
% HCONTROL : = 0: use constant step size H0
% = 1: use step size control
% = 2: use clustered step size control
% 
% Output parameters:
% T : vector with points of independent variable
% Y : matrix with approximations at points t
% ISTAT : integration statistics
% ISTAT = [number of accepted steps,
% number of rejected steps,
% number of function evaluations,
% number of pcg evaluations];
% number of pcg errors];
% IDID : = 1: integrator has reached tend
% = -1: abort (step size too small)
% = -2: abort (solver did not converge)
% 
% Set a global variable ‘verbose’ to a file id (‘fid’) 
% to see or save debug information
```
Listing C.6: Gauß-Seidel. Matlab m-file implementing the Gauß-Seidel scheme with constant or dynamic controlled iterations

```matlab
function [t,y,istat,idid] = gaussseidel(fitmodel,yfit0,...
    mnamodel,ymna0,...
    tspan,rtol,atol,H,l)

% GAUSSSEIDEL Solves the FIT/MNA coupled system
% with the Gauss-Seidel approach.

% [T,Y,ISTAT,IDID] = GAUSSSEIDEL(MCPL,KCPL,QCPL,YFIT0,...
% MMNA,KMNA,QMNA,YMNA0,SRCFUN...
% SHRINK,REPLACE,INTEREST,...
% TSPAN,RTOL,ATOL,H,LMIN)

% Input parameters:
% MFIT : constant mass matrix of the FIT system
% KFIT : constant curl-curl matrix of the FIT system
% QFIT : constant right-hand side matrix of the FIT system
% YFIT0 : intial value FIT
% MMNA : constant mass matrix of the MNA system
% KMNA : constant stiffness matrix of the MNA system
% QMNA : constant right-hand side matrix of the MNA system
% YMNA0 : intial value MNA
% SRCFUN : Source functions of the MNA system (voltage/current)
% SHRINK : indeces of non zero rows/columns
% REPLACE : indeces of rows/columns for the BC
% INTEREST : indeces of fit variables that are exchanged
% TSPAN : vector [T0,TEND]
% RTOL,ATOL : relative and absolute tolerance
% H : communication step size ("macro step size")
% L : = -1: Adaptive control of window iterations (min. 2)
%    = 0: No iteration of time windows (multi-rate co-sim)
%    >0: Constant number of iterations for all time windows

% Output parameters:
% T : vector with points of independent variable
% Y : matrix with approximations at points t
% ISTAT : integration statistics
%     ISTAT = [number of macro steps,
%              number of micro steps
%              number of iterations];
% IDID : = 1: integrator has reached tend
%        = -1: abort (solver did not converge)
%        = -2: abort (Too many iterations of a window)

Set a global variable 'verbose' to a file id ('fid')
to see or save debug information
```
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