Portfolio Optimization

The Martingale Approach

Master Thesis

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May 2006

To my family.

I thank Alexandra Claßen for her love, patience and understanding. Special thanks go to my supervisor Prof. Dr. Michael Günther, Prof. Dr. Manfred Mendel, Christian Kahl and Florian Unkel for their support and the time they spent for me.

Patrick Deuß, May 2006

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Abbreviations

Please note that all vectors in this thesis are defined as column vectors.

1. $a \wedge b := \min\{a, b\}$
2. $\mathbb{R}^+ := \{x \in \mathbb{R} | x > 0\}$
3. $\mathbb{R}_0^+ := \{x \in \mathbb{R} | x \geq 0\}$
4. $e^n := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$
5. w.l.o.g. := without loss of generality
6. P-a.s. := P almost surely
7. w.r.t. := with respect to
8. P-a.e. := P almost every/everywhere
9. $f^+ := \max\{0, f\}$, $f^- := \min\{0, f\}$
10. $\mathcal{B}^n :=$ Borel-algebra on $\mathbb{R}^n$
11. $\mathcal{B}[0, t] :=$ Borel-algebra on the interval $[0, t]$
12. $\mathcal{B} :=$ Borel-algebra on $\mathbb{R}$
13. $a^t :=$ the transposed of vector a
14. $x \in \mathbb{R}^{n \times m} := \begin{pmatrix} x_{11} & \dots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nm} \end{pmatrix}$
    with $x_{ij} \in \mathbb{R}$, $i = 1, \dots, n$ and $j = 1, \dots, m$
15. $f \in \mathcal{C}^0(0, \infty) := f$ is continuous on the interval $(0, \infty)$
16. $f \in \mathcal{C}^n$, $n \in \mathbb{N} := f$ is n-times continuously differentiable
17. $f \in \mathcal{C}^{n,m}$, $n, m \in \mathbb{N} := f$ is n-times continuously differentiable in the first variable and m-times continuously differentiable in the second one
18. s.t. := such that
19. $a \vee b := a$ or b
20. $\lim\limits_{x \downarrow a} f(x) := \lim\limits_{n \in \mathbb{N}} f(a + \frac{1}{n})$ as right-hand limit
21. p.e. := for example

Introduction

'How can we optimize our portfolio?' For sure, there are many ways to find solutions for this question. One mathematical attempt is presented in this thesis: portfolio optimization via the martingale approach.

What do we understand by a portfolio? Normally, we declare a portfolio as a compound of all conceivable instruments in the financial market (bonds, stocks, options, funds, all kinds of derivates, real estates, certificates, commodities etc. - the list could be amplified arbitrarily).
To simplify the calculus and formula in this thesis we confine our portfolio to stocks and bonds. Please note that almost all other kinds of securities are either similar to stocks or can be replicated by them in an analytical-mathematical way (compare the strategy of duplication - option pricing).

Portfolio optimization and its history

The first one who made investment behavior more comprehensive in an analytical way was Harry Markowitz with his article 'Portfolio Selection' in the 'Journal Of Finance', 1952 (Markowitz [1]). This ground-breaking work was the premier that showed that risk / volatility and expectation of securities can be understood as statistical measures, namely as variance and as expected value and it strongly influenced the portfolio management (the capital asset pricing model=CAPM is based on Markowitz' explanation report) and portfolio optimization.

Markowitz indicates how to compute efficient portfolios. In a mathematical sense, this computation is a calculation of optimal surfaces by linear or non-linear programming.
An efficient portfolio is one 'with minimum risk for given expectation' or one 'with maximum expectation and given risk'. Markowitz' model is a discrete-time one. The subsequent approaches for the portfolio problem, whether in a discrete-time or in a continuous-time market model, use deeper mathematics such as the 'stochastic control approach' introduced by Merton [12, 13] in 1969 or the 'martingale approach' in different versions (1980ies), for example.

The problem

An investor is endowed with a certain starting capital which he has to allocate on his portfolio, the 30 DAX corporations and one riskless bond, for instance. During his investment (a finite time horizon) he can consume and redistribute the shares of the securities. This means we search for a continuous-time solution.

His aim is to maximize the utility of consumption and the terminal wealth of his portfolio. How the investor chose the utility depends on himself.
Furthermore, we suppose that the necessary data (expected returns, volatilities and correlations etc.) are given by the former performances and observations and the near future expectations of the assets and are available for the investor.

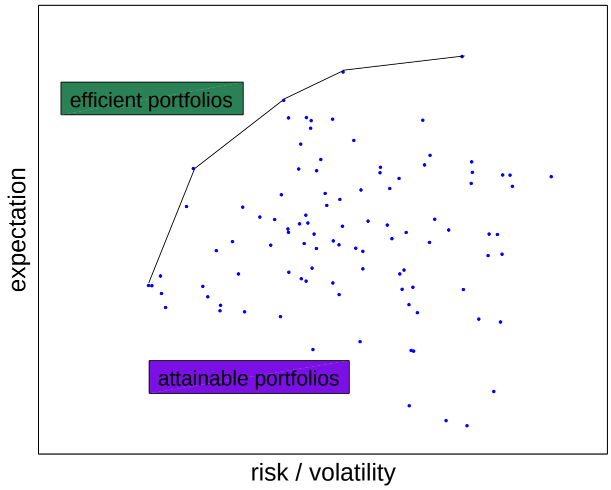

Figure 1: Efficient portfolios according to Markowitz

<u>One solution</u>

This thesis presents one solution of the portfolio problem: the martingale approach. In our methodology we will follow predominantly Korn [2, 4].

We assume that our market is complete in a mathematical and economical sense (this will be explained later, see chapter 2) which is evident for this approach. Moreover, the solution requires a continuous-time market model.

Concerning the composition of this thesis it must be pointed out that the reader is completely introduced to the subject of portfolio optimization: special previous knowledge is not needed with the exception of mathematical basics. He might find some helpful comments in appendix A, long proofs are attached in appendix B. Please note the abbreviations at the very beginning.

In chapter 1 basic terms and tools of stochastic analysis such as stochastic differential equations or the Itô calculus are presented and provide a basis for chapter 2 and 3. The second chapter deals with the modelling of the *complete* financial market. Here, stock prices are created, for example. In chapter 3 the derivation of the martingale approach and a general solution of the portfolio problem are described. We apply the obtained solutions to a portfolio of the 30 DAX corporations and one bond (the initial problem) in chapter 4 and evaluate this numerical example. A conclusion is given in chapter 5.

1 Stochastic analysis

In this chapter the mathematical basis will be created to handle the problems we will be faced with during this thesis. The composition should be self-explanatory. In addition to that the reader might find some helpful annotations and explications in the appendix.

1.1 Stochastic processes and martingales

If we want to model a financial market, we have to think about its basic instruments and properties. First, we need a sort of 'time structure' connected with a description of the 'white-noise-effect' or chance / random in the market.

Assume that $(\Omega, \mathcal{F}, P)$ is a complete probability space.

1.1 Definition
Let $\{\mathcal{F}_t\}_{t\in I}$ be a family of sub-σ-algebras of $\mathcal{F}$ and $I \subset \mathbb{R}$ an interval with $I := [0,T]$, $T \in \mathbb{R}^+$, ($T = \infty$ possible: $I = [0,\infty)$).
If $\mathcal{F}_s \subseteq \mathcal{F}_t$ is held for $s \leq t$, $s,t \in I$, then $\{\mathcal{F}_t\}_{t\in I}$ is a ***filtration***.

1.2 Definition
A set $\{(X_t, \mathcal{F}_t)\}_{t\in I}$ such that

i) $\{\mathcal{F}_t\}_{t\in I}$ is a filtration

ii) $\{X_t\}_{t\in I}$ is a family of random variables assuming values in $\mathbb{R}$ in which X_t is $\mathcal{F}_t$-measurable with
$X : [0,T] \times \Omega \to \mathbb{R}$
$(t,\omega) \mapsto X_t(\omega) := X(t,\omega)$

is called ***stochastic process*** with filtration $\{\mathcal{F}_t\}_{t\in I}$.

We remark that

i) for fixed $t \in I$, $\omega \in \Omega$, $X_t(\omega)$ with $\omega \mapsto X_t(\omega)$ is a random variable,

ii) for fixed $\omega \in \Omega$, $t \in I$, $X_t(\omega)$ with $t \mapsto X_t(\omega)$ is a function depending only on time t. We call this ***path*** or ***realisation*** of a stochastic process.

iii) Abbreviation: $X_t := X(t,\omega) := \{(X_t, \mathcal{F}_t)\}_{t\in I}$

Explanation for definitions 1.1 and 1.2

The σ-algebra $\mathcal{F}_t$, $t \in I$, will model the observable occurrences of a random variable X_t till time t. If X_t is $\mathcal{F}_t$-measurable, we can determine the values of X_t at time t. Therefore, a filtration $\{\mathcal{F}_t\}_{t\in I}$ reflects a time structure.
A stochastic process will be an elementary instrument to formalize random depending on time and the market situation.

As we will see later, one example of a stochastic process plays an important role in modelling stock prices: the Brownian motion.

A one-dimensional Brownian motion $B_t(\omega)$ is a stochastic process $\{B_t\}_{t\in I}$ with the following properties:

i) $B_t(\omega) \in \mathbb{R} \; \forall \; t \in I, \; \omega \in \Omega, \; B : [0,T] \times \Omega \mapsto \mathbb{R}$

ii) $B_t(\cdot) \in \mathcal{C}^0 \; \forall \; \omega \in \Omega$ (i.e. continuous paths for fixed $\omega \in \Omega$) and

  a) $B_0(\omega) = 0$ P-a.s.

  b) $B_t(\omega) - B_s(\omega) \sim N(0, t-s)$ for $0 \leq s \leq t$ ***(stationary increases)***

  c) $B_t(\omega) - B_s(\omega)$ is independent from $B_u(\omega) - B_r(\omega)$
  for $0 \leq r \leq u \leq s \leq t$ ***(independent increases)***

For the n-dimensional case we have $B_t(\omega) = (B_1(t,\omega), \ldots, B_n(t,\omega))^t$ such that $B_i(t,\omega)$ are independent, identically distributed one-dimensional Brownian motions.

The filtration for the Brownian motion can be defined on the one hand as

$$\overline{\mathcal{F}_t^B} := \sigma\{B_s | 0 \leq s < t\} \text{ \textbf{\textit{(natural filtration)}}}$$

or, on the other hand as

$$\mathcal{F}_t^B := \sigma\left\{\overline{\mathcal{F}_t^B} \cup N | N \in \mathcal{F}, P(N) = 0\right\} \text{ \textbf{\textit{(Brownian filtration)}}}.$$

Because of some technical reasons we will use the Brownian filtration. To prove that the Brownian motion exists as a stochastic process is very technical. For further information see Billingsley [15].

We note that the Brownian filtration $\{\mathcal{F}_t^B\}_{t\in I}$ is right- and left-continuous (see Karatzas and Shreve [16]).

- right-continuous : $\mathcal{F}_t = \mathcal{F}_{t^+} := \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$
- left-continuous : $\mathcal{F}_t = \mathcal{F}_{t^-} := \sigma\left(\bigcup_{s\leq t} \mathcal{F}_s\right)$

1.3 Definition
A filtration $\{\mathcal{F}_t\}_{t\in I}$ fulfils the ***usual conditions***, if

i) $\{\mathcal{F}_t\}_{t\in I}$ is right-continuous and

ii) $\mathcal{F}_0$ contains all sets $N \subset \Omega$ with P-outer measure zero ($P(N) = 0$).

Explanation

In the following we will use filtrations holding definition 1.3 which guarantees that our 'time structure' is adequately smooth.

How can we describe and analyze a random variable under changed conditions? Which values or expectation does it assume if we underly a different σ-algebra for example? The answer to this question is given in the next definition (compare also Øksendal [5])

1.4 Definition
Let $X : \Omega \mapsto \mathbb{R}$ be a random variable with $E[|X|] < \infty$ and $\mathcal{H} \subset \mathcal{F}$ a σ-algebra. The ***conditional expectation*** $E[X|\mathcal{H}]$ is the P-a.s. unique function such that

i) $E[X|\mathcal{H}]$ is $\mathcal{H}$-measurable and

ii) $\forall\ H \in \mathcal{H}$ and $\forall$ random variables Z, which are $\mathcal{H}$-measurable

$$\int\limits_H E[X|\mathcal{H}]dP = \int\limits_H XdP \quad \Leftrightarrow \quad \int\limits_\Omega Z \cdot E[X|\mathcal{H}]dP = \int\limits_\Omega Z \cdot XdP$$

is held.

Please note that uniqueness and existence of $E[X|\mathcal{H}]$ are deduced from the theorem of Radon-Nikodym (see Bauer [6] or Michel [10]). For the calculus with $E[X|\mathcal{H}]$ we take a closer look at the next theorem.

1.5 Theorem
Properties of the conditional expectation $E[X|\mathcal{H}]$ are

i) $E[E[X|\mathcal{H}]] = E[X]$

ii) $E[X|\mathcal{H}] = X$ if X is $\mathcal{H}$-measurable

iii) $E[X|\mathcal{H}] = E[X]$ if X is independent of $\mathcal{H}$.

Without proof.

We will learn that one class of stochastic processes are well suited for the mathematical depiction of the financial market: martingales.

1.6 Definition
Let $\{(X_t, \mathcal{F}_t)\}_{t\in I}$ be a stochastic process with $E[|X_t|] < \infty\ \forall\ t \in I$.
If $E[X_t|\mathcal{F}_s] = X_s$ P-a.s. $\forall\ s,t \in I,\ s \le t$, $\{(X_t, \mathcal{F}_t)\}_{t\in I}$ is called ***martingale***.
The process X_t with

a) $E[X_t|\mathcal{F}_s] \le X_s$ is a ***supermartingale***,

b) $E[X_t|\mathcal{F}_s] \ge X_s$ is a ***submartingale***.

Explanation

Let X_n, $n \in \mathbb{N}$, be the amount of money an investor has after n trading days on the market. Then a 'fair' market would fulfil the martingale condition $E[X_{n+1}|\mathcal{F}_n] = X_n$ P-a.s.. That means that the *expected* amount of money after the $(n+1)$-th participation on the market is the same as after n-days. A favorable market for the investor would be a submartingale ($E[X_{n+1}|\mathcal{F}_n] \ge X_n$), an inconvenient market a supermartingale ($E[X_{n+1}|\mathcal{F}_n] \le X_n$).

1.7 Corollary
The one-dimensional Brownian motion B_t is a martingale.

Proof

$$\begin{aligned} E[B_t|\mathcal{F}_s] &= E[B_t - B_s + B_s|\mathcal{F}_s] \\ &= E[B_t - B_s|\mathcal{F}_s] + E[B_s|\mathcal{F}_s] && (\curvearrowright B_s \text{ is } \mathcal{F}_s\text{-measurable}) \\ &= E[B_t - B_s|\mathcal{F}_s] + B_s && (\curvearrowright B_t - B_s \text{ are independent of } \mathcal{F}_s) \\ &= E[B_t - B_s] + B_s && (\curvearrowright B_t - B_s \sim N(0, t-s)) \\ &= B_s && \blacksquare \end{aligned}$$

1.8 Example
Let $X_i(t) = \mu_i \cdot t + \sigma \cdot B_t$, μ_i, $\sigma \in \mathbb{R}$, $t \geq 0$, $i = 1, 2, 3$, be a Brownian motion with *drift* μ and *volatility* σ. Then we have

i) $\mu_1 > 0 \Rightarrow X_1(t)$ is a submartingale,

ii) $\mu_2 = 0 \Rightarrow X_2(t)$ is a martingale,

iii) $\mu_3 < 0 \Rightarrow X_3(t)$ is a supermartingale.

In the next figure the different drifts are defined by $\mu_1 = 1$, $\mu_2 = 0$, $\mu_3 = -1$ whereas the volatilities are equal for $X_i(t)$, $i = 1, 2, 3$, that is to say $\sigma = 1$.

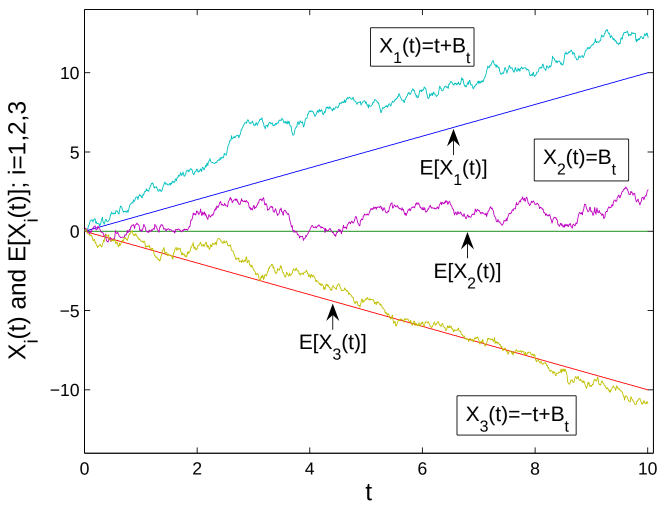

Figure 1.1: Brownian motions and their expectations

1.2 The Ito calculus

Later, when we will model stock prices, the question will occur how to solve a stochastic integral such as

$$\int_0^t X_s(\omega)dB_s(\omega) \tag{1.1}$$

in which $B_t(\omega)$ is a Brownian motion and $X_t(\omega)$ a measurable, non-negative function with values in $\mathbb{R}$, $X_t(\omega) : [0,T] \times \Omega \mapsto \mathbb{R}_0^+$. We assume that $\mathcal{F}_t$ is a Brownian filtration fulfilling the usual conditions. Still $(\Omega, \mathcal{F}, P)$ is a complete probability space.

The Ito integral

To make a long story short, we have to realize that we cannot compute (1.1) in the Riemann-Stieltjes sense:
If μ is a distribution for Y, $\mu \in \mathcal{C}^1$, with density $\nu(s) = \frac{d\mu(s)}{ds}$, we solve the Riemann integral

$$\int_0^t Y(s)d\mu(s) = \int_0^t Y(s)\nu(s)ds.$$

This approach is not possible for (1.1) because the realisations of a Brownian motion $B_t(\omega)$ are P-almost nowhere differentiable. Furthermore, we cannot define (1.1) as a Lebesgue-Stieltjes integral; the paths of B_t have P-a.e. infinite variations.
Hence, we need some sort of approximation for (1.1). The idea is the following: we will define a stochastic integral for elementary processes. Then we will show that stochastic processes (with certain restrictions) can be approximated by these elementary processes and we will finally find a solution for (1.1).

1.9 Definition
A stochastic process $\phi_t^{(n)}(\omega)$ is an ***elementary process***, if

i) $\exists\ t_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, $n \in \mathbb{N}$, with $0 = t_0 < t_1 < \ldots < t_n = T$

ii) $\exists$ bounded random variables $\varphi_i(\omega)$, $i = 0, \ldots, n$, $|\varphi_i| < \infty$, $\varphi : \Omega \mapsto \mathbb{R}$ with

  a) φ_0 is $\mathcal{F}_0$-measurable

  b) φ_i is $\mathcal{F}_{t_{i-1}}$-measurable

such that

$$\phi_t^{(n)}(\omega) = \varphi_0(\omega)\chi_0(t) + \sum_{i=1}^n \varphi_i(\omega)\chi_{(t_{i-1},t_i]}(t) \qquad \forall\ \omega \in \Omega$$

in which $\chi_{(t_{i-1},t_i]}(t) := \begin{cases} 1 \text{ if } t \in (t_{i-1}, t_i] \\ 0 \text{ otherwise} \end{cases}$

Note that

- χ is the indicator function,
- $\phi_t^{(n)}(\omega)$ is $\mathcal{F}_{t_{i-1}}$-measurable for $t \in (t_{i-1}, t_i]$,
- the more $n \in \mathbb{N}$ grows, the smaller the intervals $(t_{i-1}, t_i]$ get. This means that with $n \to \infty$ the interval $I = [0, T]$ is divided into infinite parts, so to say: $(t_{i-1}, t_i] \to t$ as $n \to \infty$.

Notation
$\mathcal{S} := \{X_t | X_t$ is an elementary process$\}$.

1.10 Example
A path of an elementary process $\phi_t^{(n)}$ is a left-continuous step function.

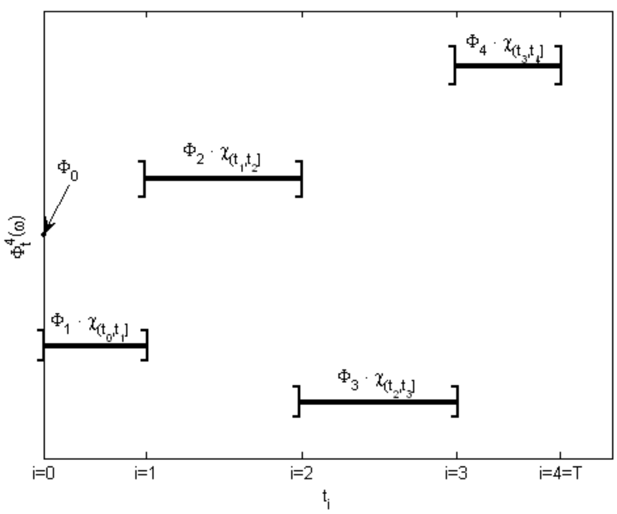

Figure 1.2: A left-continuous step function ϕ_t^4 with $\phi_i = \varphi_i$, $i = 1, \ldots, 4$

Now, we define (1.1) for elementary processes.

1.11 Definition
Let $\phi_t^{(n)} \in \mathcal{S}$, $t \in I$.

The ***stochastic integral*** $I_\bullet[\phi]$ is defined for $t \in I$ as

$$I_t\left[\phi^{(n)}\right](\omega) := \int_0^t \phi_s^{(n)}(\omega) dB_s(\omega) := \sum_{i=1}^{n} \varphi_i(\omega) \underbrace{\left[B_{t_i \wedge t}(\omega) - B_{t_{i-1} \wedge t}(\omega)\right]}_{:= \Delta}$$

Interpretation and remarks

1) For $t \in (t_{i-1}, t_i]$ it is $\phi_t^{(n)} \equiv \varphi_i$, $\forall\, \omega \in \Omega$.
2) Δ is the increase of the Brownian motion B_t on $(t_{i-1}, t_i]$.
3) We multiply the increases ($\triangleq \Delta$) by the values of $\phi_t^{(n)} (\triangleq \varphi_i)$.
4) For $\phi_t, \theta_t \in \mathcal{S}$, $a, b \in \mathbb{R}$ we have

$$I_t[a\phi_t + b\theta_t](\omega) = aI_t[\phi_t](\omega) + bI_t[\theta_t](\omega)$$

5)

$$\int_r^t \phi_s(\omega)dB_s(\omega) = \int_0^t \phi_s(\omega)dB_s(\omega) - \int_0^r \phi_s(\omega)dB_s(\omega)$$

For the approximation of elementary processes to stochastic processes in general we need a new term concerning the measurability of processes.

1.12 Definition
Let X_t be a stochastic process. Then X_t is $\mathcal{F}_t$***-adapted*** if

$$\begin{aligned} \overline{X} : [0,t] \times \Omega &\to \mathbb{R}^n \\ (s,\omega) &\mapsto X_s(\omega) \end{aligned}$$

is $\mathcal{B}[0,t] \otimes \mathcal{F}_t - \mathcal{B}^n$ measurable $\forall\, t \in I$.

Remember that $\{\mathcal{F}_t\}_{t\in I}$ is an increasing family of σ-algebras in Ω.

Notation
Let $\mathcal{L}^2[0,T] := \mathcal{L}^2([0,T], \Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, P)$ be the class of functions of

$X_t(\omega) = X(t,\omega) : [0,T] \times \Omega \to \mathbb{R}$ such that

i) $\overline{X} : (t,\omega) \mapsto X(t,\omega)$ is $\mathcal{B} \otimes \mathcal{F}$-measurable
ii) $X(t,\omega)$ is $\mathcal{F}_t$-adapted and
iii) $E\left[\int_0^T X_t^2(\omega)dt\right] < \infty$ (i.e. is bounded).

Note that $\mathcal{L}^2[0,T]$ is a vector space and that $\{(X_t, \mathcal{F}_t)\}_{t\in I}$ is a $\mathbb{R}$-valued stochastic process.
Now, we want to approximate $X \in \mathcal{L}^2[0,T]$ by $\phi^{(n)} \in \mathcal{S}$, $n \in \mathbb{N}$, in other words:

$$\lim_{n\in\mathbb{N}} \phi^{(n)} \to X$$

With this observation we will show that $I[X] = \lim_{n\in\mathbb{N}} I\left[\phi^{(n)}\right]$ is held thereby resolving our problem for (1.1) for $X \in \mathcal{L}^2[0,T]$. First, we introduce an important property of $\phi \in \mathcal{S}$.

1.13 Lemma (The Itô isometry)
For $\phi \in \mathcal{S}$, ϕ bounded, it is

$$E\left[\left(\int_0^t \phi^{(p)}(s,\omega)dB_s(\omega)\right)^2\right] = E\left[\int_0^t \phi^{(p)}(s,\omega)^2 dt\right] \quad \forall\, t \in [0,T] \qquad (1.2)$$

Proof: see appendix B.
Use the definition of ϕ and compute $E\left[I_t(\phi)^2\right]$ with help of a case differentiation ($i \neq j$ and $i = j$).

The result of the Itô isometry is of course very useful, because we do not need to integrate in dependence of the Brownian motion $B_t(\omega)$. Instead, we calculate the integral in dependence of time t. Vice versa we know then that $E\left[\left(\int XdB_s\right)^2\right]$ exists.

With the next fundamental theorem we will show that an approximation as we mentioned above can be found.

1.14 Theorem
Let $X \in \mathcal{L}^2[0,T]$. Then $\exists$ a sequence $\{\phi^{(n)}\}_{n\in\mathbb{N}}$, $\phi^{(n)} \in \mathcal{S}$ $\forall\, n \in \mathbb{N}$ such that

$$\lim_{n\in\mathbb{N}} E\left[\int_0^T \left(X_t - \phi_t^{(n)}\right)^2 dt\right] = 0 \qquad (1.3)$$

Proof: see appendix B.
Write X_t as elementary process ϕ_t^n with a suited indicator function. Show that this process converges to X_t as $n \to \infty$ with help of theorem of dominated convergence A.4, appendix A.2.

1.15 Definition (The Itô integral)
Let $X \in \mathcal{L}^2[0,T]$, $\phi^{(n)} \in \mathcal{S}$, $n \in \mathbb{N}$, with $E\left[\int_0^T \left(X_t(\omega) - \phi_t^{(n)}(\omega)\right)^2 ds\right] \to 0$ as $n \to \infty$. Then $I_t[X](\omega)$ is defined as

$$I_t[X](\omega) := \int_0^t X_s(\omega)dB_s(\omega) := \lim_{n\in\mathbb{N}} \int_0^t \phi^{(n)}(s,\omega)dB_s(\omega) \qquad (1.4)$$

as limit in $\mathcal{L}^2[0,T]$.

Remark

i) $I_t[X](\omega)$ exists (i.e. the limit (1.4) exists) and is independent of the choice of $\{\phi^{(n)}\}_{n\in\mathbb{N}}$ as long as (1.3) is held.

ii)

$$\int_0^t \phi^{(n)}(s,\omega)dB_s(\omega) = \sum_{i=1}^{n} \varphi_i(\omega)\left[B_{t_i\wedge t}(\omega) - B_{t_{i-1}\wedge t}(\omega)\right]$$

$$\Rightarrow \qquad I_t[X](\omega) = \lim_{n\in\mathbb{N}} \sum_{i=1}^{n} \varphi_i(\omega)\left[B_{t_i\wedge t}(\omega) - B_{t_{i-1}\wedge t}(\omega)\right]$$

iii) The sequence of elementary processes $\{\phi^{(n)}\}_{n\in\mathbb{N}}$ exists, proven by theorem (1.14).

1.16 Corollary (The Itô isometry for functions in $\mathcal{L}^2[0,T]$)
Let $X \in \mathcal{L}^2[0,T]$. Then

$$E\left[\left(\int_0^T X_t(\omega)dB_t(\omega)\right)^2\right] = E\left[\int_0^T X_t(\omega)^2 dt\right]$$

is valid.

Proof
Follows from equations (1.2) and (1.4). ∎

1.17 Corollary
Let $X_t \in \mathcal{L}^2[0,T]$, $\phi_t^{(n)} \in \mathcal{S}$, $n \in \mathbb{N}$, with $E\left[\int_0^T \left(\phi_t^{(n)} - X_t\right)^2 dt\right] \to 0$ as $n \to \infty$.

$$\Rightarrow \quad \lim_{n\in\mathbb{N}} \int_0^T \phi_t^{(n)} dt = \int_0^T X_t dt \quad \text{in } \mathcal{L}^2[0,T]$$

Without proof.

To illustrate the calculus with stochastic or Itô integrals, we regard the next example.

1.18 Example

$$\int_0^T B_s(\omega)dB_s(\omega) = \frac{1}{2}B_T^2(\omega) - \frac{T}{2}$$

$$\int_0^T 1 dB_s(\omega) = B_T$$

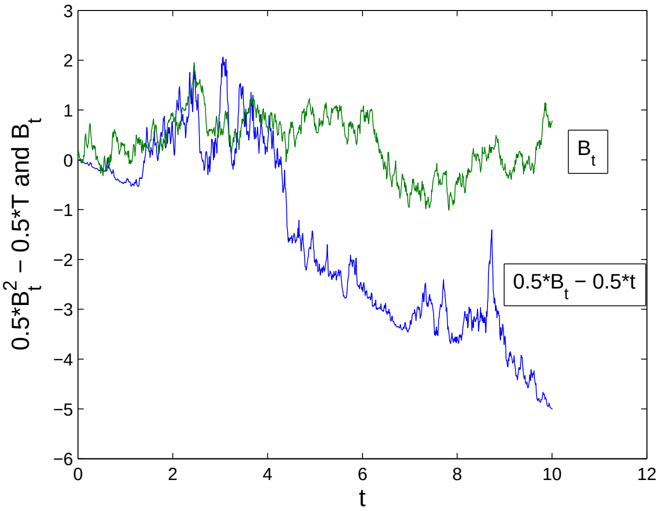

Figure 1.3: Possible representations for $\frac{1}{2}B_t^2(\omega) - \frac{t}{2}$ and $B_t(\omega)$

Proof

$\phi_s^{(n)} := \sum_{k=1}^{n} B_{t_k} \cdot \chi_{(t_{k-1},t_k]}(s) \quad \Rightarrow \quad \lim_{n\in\mathbb{N}} \phi_s^{(n)} = B_s$

$$
\begin{aligned}
& E\left[\int_0^T \left(\phi_s^{(n)} - B_s\right)^2 ds\right] \\
=& E\left[\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \left(\phi_s^{(n)} - B_s\right)^2 ds\right] \qquad (\curvearrowright \phi_s^{(n)} \equiv B_{t_k} \text{ for } s \in (t_{k-1}, t_k]) \\
=& \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} E\left[(B_{t_k} - B_s)^2\right] ds = \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} Var(B_{t_k} - B_s) ds \\
=& \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} (t_k - s) ds = \sum_{k=1}^{n}\left[t_k \cdot s - \frac{1}{2}s^2\right]_{t_{k-1}}^{t_k} \\
=& \frac{1}{2}\sum_{k=1}^{n}(t_k - t_{k-1})^2 \to 0 \quad (\text{as } n \to \infty, \text{ because } t_k \to t_{k-1}\ )
\end{aligned}
$$

This means as $n \to \infty$(compare corollary 1.16 and definition 1.15)

$$
E\left[\left(\int_0^T (\phi_s^{(n)} - B_s) dB_s\right)^2\right] \to 0 \quad \Rightarrow \quad \int_0^T \phi_s^{(n)} dB_s \to \int_0^T B_s dB_s
$$

Furthermore, we have

$$\begin{aligned}
&\sum_{k=1}^{n} B_{t_k}(B_{t_k}-B_{t_{k-1}}) = \sum_{k=1}^{n} B_{t_k}^2 - \sum_{k=1}^{n} B_{t_k}B_{t_{k-1}} \qquad (\curvearrowright B_{t_n}=B_T)\\
&= \frac{1}{2}B_T^2 + \frac{1}{2}\sum_{k=1}^{n} B_{t_k}^2 + \frac{1}{2}\sum_{k=1}^{n-1} B_{t_k}^2 - \sum_{k=1}^{n} B_{t_k}B_{t_{k-1}} \qquad (\curvearrowright B_{t_0}=B_0=0)\\
&= \frac{1}{2}B_T^2 + \frac{1}{2}\sum_{k=1}^{n} B_{t_k}^2 + \frac{1}{2}\sum_{k=1}^{n} B_{t_{k-1}}^2 - \sum_{k=1}^{n} B_{t_k}B_{t_{k-1}}\\
&= \frac{1}{2}B_T^2 + \frac{1}{2}\sum_{k=1}^{n} \left(B_{t_k}-B_{t_{k-1}}\right)^2
\end{aligned}$$

With this we reach for

$$\begin{aligned}
&\lim_{n\in\mathbb{N}} \int_0^T \phi_s^{(n)} dB_s \qquad (\curvearrowright \text{definition 1.11})\\
=\ &\lim_{n\in\mathbb{N}} \sum_{k=1}^{n} B_{t_k}(B_{t_k}-B_{t_{k-1}})\\
=\ &\frac{1}{2}B_T^2 + \frac{1}{2}\underbrace{\lim_{n\in\mathbb{N}} \sum_{k=1}^{n}(B_{t_k}-B_{t_{k-1}})^2}_{:=\eta} \qquad (\eta \to T \text{ in } \mathcal{L}^2[0,T], \text{ without proof})\\
\Rightarrow\ &\int_0^T \phi_s^{(n)} dB_s = \frac{1}{2}B_T^2 + \frac{1}{2}T
\end{aligned}$$

And, for the other equation:

$$\int_0^T 1 dB_s = \sum_{k=1}^{n}(B_{t_k}-B_{t_{k-1}}) = \sum_{k=1}^{n} B_{t_k} - \sum_{k=1}^{n} B_{t_{k-1}} = B_{t_n} = B_T$$

■

For the generalization from a one-dimensional Itô integral to a multi-dimensional one we have:

Notation

Let $X(t,\omega) := \begin{pmatrix} X_{11}(t,\omega) & \dots & X_{1m}(t,\omega) \\ \vdots & \ddots & \vdots \\ X_{n1}(t,\omega) & \dots & X_{nm}(t,\omega) \end{pmatrix}$, $\{(X_t,\mathcal{F}_t)\}_{t\in I}$ be a $\mathcal{F}_t$-adapted process with $X_{ij} \in \mathcal{L}^2[0,T]$ assuming values in $\mathbb{R}^{n\times m}$ and filtration $\{\mathcal{F}_t\}_{t\in I}$, $\{(B(t,\omega),\mathcal{F}_t)\}_{t\in I}$ a m-dimensional Brownian motion with $B(t,\omega) = (B_1(t,\omega),\dots,B_m(t,\omega))^t$.

Then we define

$$\int_0^t X(s,\omega)dB_s(\omega) := \begin{pmatrix} \sum_{j=1}^{m} \int_0^t X_{1j}(s,\omega)dB_j(s,\omega) \\ \vdots \\ \sum_{j=1}^{m} \int_0^t X_{nj}(s,\omega)dB_j(s,\omega) \end{pmatrix}$$

Because of lemma 1.7 we know that $\sum_j \int X_{ij}dB_j$ still remains a martingale, moreover the single summands are one-dimensional Itô integrals.

Continuation from $\mathcal{L}^2[0,T]$ to $\mathcal{H}^2[0,T]$

We want to enlarge the class of processes from $\mathcal{L}^2[0,T]$ to a greater vector space $\mathcal{H}^2[0,T]$. This allows us to apply our instruments to a bigger class of stochastic processes than before.

Notation

$$\mathcal{H}^2[0,T] := \mathcal{H}^2\left([0,T], \Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, P\right)$$

$$:= \left\{ \{(X_t, \mathcal{F}_t)\}_{t\in I}, X_t(\omega) \in \mathbb{R} | \{X_t\}_{t\in I} \text{ is } \mathcal{F}_t\text{-adapted}, \int_0^T X_t^2 dt < \infty \ P\text{-a.s.} \right\}$$

$X \in \mathcal{H}^2[0,T]$ cannot be approximated by $\phi^{(n)} \in \mathcal{S}$ (elementary processes) as we did for processes in $\mathcal{L}^2[0,T]$. But they can be located by stopping times τ and with this sort of location we achieve that $\overline{X} := X|\tau \in \mathcal{L}^2[0,T]$. The approximation for $\overline{X}$ follows analogous as for all processes in $\mathcal{L}^2[0,T]$.

1.19 Definition
A $\mathcal{F}$-measurable function $\tau : \Omega \to [0,T]$ w.r.t. a filtration $\{\mathcal{F}_t\}_{t\in I}$ with $A_t := \{\omega \in \Omega | \tau(\omega) \leq t\} \in \mathcal{F}_t \ \forall \ t \in I$ is called ***stopping time***.

Explanation
With help of a stopping time it is possible to stop a stochastic process and hold it in its actual condition.

$X_{t\wedge\tau}(\omega) := \begin{cases} X_t(\omega) \text{ for } t \leq \tau \\ X_{\tau(\omega)}(\omega) \text{ for } t > \tau \end{cases}$ is a ***stopped process***.
If we define a new σ-algebra, an 'algebra of occurrences', till time τ as $\mathcal{F}_\tau := \{A \in \mathcal{F} | A \cap A_t \in \mathcal{F}_t \ \forall \ t \in I\}$, we will get the ***stopped filtration*** $\{\mathcal{F}_{t\wedge\tau}\}_{t\in I}$. Note that τ is $\mathcal{F}_\tau$-measurable and $\mathcal{F}_{t\wedge\tau} \subset \mathcal{F}_t$.

The restriction $A_t \in \mathcal{F}_t$ tells us that we are able to decide whether or not to stop the process. An example for a stopping time would be an investor who sells all his securities when they have reached a certain value.

These stopping times enable us to 'locate' stochastic processes as we see in the next definition. Let be $I := [0, \infty)$.

1.20 Definition
Let $\{(X_t, \mathcal{F}_t)\}_{t\in I}$ be a stochastic process with $X_0 = 0$. If $\exists \ \{\tau_n\}_{n\in\mathbb{N}}$ with $\tau_n \leq \tau_{n+1}$ a sequence of stopping times with $P\left(\lim\limits_{n\in\mathbb{N}} \tau_n = \infty\right) = 1$, such that $\left\{X_t^{(n)} := X_{t\wedge\tau_n}, \mathcal{F}_t\right\}_{t\in[0,\infty)}$ is a martingale $\forall \ n \in \mathbb{N}$, then we call X_t a ***local martingale*** and $\{\tau_n\}_{n\in\mathbb{N}}$ is the ***localizing sequence***.

Now, we can localize processes $X \in \mathcal{H}^2[0,T]$: Define a stopping time $\tau_n(\omega)$ as $\tau_n(\omega) := T \wedge \inf\left\{0 \leq t \leq T | \int\limits_0^t X_s^2(\omega)ds \geq n\right\}$
and a stopped process $X_t^{(n)}(\omega)$ as $X_t^{(n)}(\omega) := X_t(\omega) \cdot \chi_{\{\tau_n(\omega)\geq t\}}(t)$

It follows (without proof): $X_t^{(n)}(\omega) \in \mathcal{L}^2[0,T]$.
Then we denote the stochastic integral as $I_t[X] := I_t\left[X^{(n)}\right]$ for $0 \leq t \leq \tau_n$.
Note that $I_t[X]$ is a local martingale by construction.

Conclusion
With this approximation from $\mathcal{H}^2[0,T]$ over $\mathcal{L}^2[0,T]$ by elementary processes we can solve the stochastic integral (1.1) for all processes in $\mathcal{H}^2[0,T]$. Thus, we presume that our processes belong to the much greater class of $\mathcal{H}^2[0,T]$.

The Itô formula

As we have seen solving stochastic integrals is not that easy. We need a significant tool which shows us a solution for some sorts of these integrals: the Itô formula. Before we reach at this theorem, we need some basic notations and definitions.

Assumptions

- $(\Omega, \mathcal{F}, P)$ is a complete probability space, $I = [0,\infty)$
- $\{\mathcal{F}_t\}_{t\in I}$ is a filtration fulfilling the usual conditions 1.3
- $\{(B_t, \mathcal{F}_t)\}_{t\in I}$ is a m-dimensional Brownian motion
- $\{K_t\}_{t\in I}$ is a $\mathcal{F}_t$-adapted process with $\int\limits_0^t |K_s| < \infty$ P-a.s. $\forall \ t \in I$
  $\Rightarrow K_t \in \mathcal{H}^2[0,T]$
- $\{L_t\}_{t\in I}$ is a $\mathcal{F}_t$-adapted, m-dimensional process with
  $L_t(\omega) := (L_1(t,\omega), \dots, L_m(t,\omega))$ and $\int\limits_0^t L_j^2(t,\omega) < \infty$ P-a.s. $\forall \ t \in I$ and
  $j = 1, \dots, m \qquad \Rightarrow \qquad L_j(t,\omega) \in \mathcal{H}^2[0,T]$

1.21 Definition
$\{(X_t, \mathcal{F}_t)\}_{t\in I}$ is a ***Itô process*** assuming values in $\mathbb{R}$,
if $\exists$ a depiction for X_t $\forall$ $t \in I$ such as

$$\begin{aligned} X_t(\omega) &= X_0(\omega) + \int_0^t K_s(\omega)ds + \int_0^t L_s(\omega)dB_s(\omega) \\ &= X_0(\omega) + \int_0^t K_s(\omega)ds + \sum_{j=1}^m \int_0^t L_j(s,\omega)dB_j(s,\omega) \end{aligned}$$

P-a.s. in which $X_0(\omega)$ is $\mathcal{F}_0$-measurable.

Notation

a) An n-***dimensional Itô process*** $X = \left(X^{(1)}, \ldots, X^{(n)}\right)^t$ has Itô processes $X^{(i)}$, $i = 1, \ldots, n$, as components.

b) For an Itô process X_t we often use the differential representation: $dX_t = K_t dt + L_t dB_t$.

c) $\mathcal{J} := \{X_t(\omega) | X_t(\omega) \text{ is Itô process }\}$

1.22 Definition
Let $X_t, Y_t \in \mathcal{J}$ with
$X_t = X_0 + \int_0^t K_s ds + \int_0^t L_s dB_s$ and $Y_t = Y_0 + \int_0^t N_s ds + \int_0^t O_s dB_s$.

$$\langle X_t, Y_t\rangle := \sum_{j=1}^m \int_0^t L_j(s,\omega)O_j(s,\omega)ds$$

is called ***quadratic covariation*** of X_t and Y_t.

$$\langle X_t\rangle := \langle X_t, X_t\rangle := \sum_{j=1}^m \int_0^t L_j^2(s,\omega)ds$$

is the ***quadratic variation*** of X_t.

Assume $X_t \in \mathcal{J}, Y_t : [0,\infty) \times \Omega \to \mathbb{R}$ is $\mathcal{F}_t$-adapted. Then we define

$$\int_0^t Y_s dX_s := \int_0^t Y_s K_s ds + \int_0^t Y_s L_s dB_s.$$

1.23 Theorem (One-dimensional Itô formula)
Let B_t be a Brownian motion, $X_t \in \mathcal{J}$ with $X_t = X_0 + \int_0^t K_s ds + \int_0^t L_s dB_s$ and $f : \mathbb{R} \to \mathbb{R}, f \in \mathcal{C}^2$ (i.e. f'' is continuous). For all $t \in I$ we have

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X_s\rangle \quad P\text{-a.s.} \\ &= f(X_0) + \int_0^t \left(f'(X_s)K_s + \frac{1}{2}f''(X_s)L_s^2\right) ds + \int_0^t f'(X_s)L_s dB_s \end{aligned} \tag{1.5}$$

Proof: see Korn [4], pp. 51-56.
First, secure with help of a localization that all expectations are defined and the marginal processes can be transposed. Applying Taylor expansion, show that the occurring sum converge to integrals of the Itô formula.

<u>Remarks</u>

a) All integrals in (1.5) are defined.

b) Once more: the differential representation for (1.5) :

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X_t\rangle$$

<u>Applications of theorem 1.23</u>

a) $X_t = t \Rightarrow X_t = 0 + \int_0^t 1ds + \int_0^t 0dB_s$

$f(X_t) = f(t)$ with $f \in \mathcal{C}^2 \Rightarrow f(t) = f(0) + \int_0^t f'(s)ds$

b) $X_t = B_t,\ f(x) = x^2,\ f(0) = 0 \Rightarrow f'(x) = 2x,\ f''(x) = 2$

$$\Rightarrow \quad B_t = 0 + \int_0^t \overset{K_s}{0}\, ds + \int_0^t \overset{L_s}{1}\, dB_s \qquad \text{(remember that } B_0 = 0\text{)}$$

$$\Rightarrow \quad B_t^2 = 2\int_0^t B_s dB_s + \frac{1}{2}\int_0^t 2ds = 2\int_0^t B_s dB_s + t \qquad \text{(cp. example 1.18)}$$

For a product of two Itô processes X_t, Y_t we can apply the Itô formula, too. The next theorem is an implication of the multidimensional Itô formula, see theorem A.8, appendix A.2.

1.24 Theorem (Partial integration)
Let $X_t, Y_t \in \mathcal{J}$ with $dX_t = K_t dt + L_t dB_t$ and $dY_t = N_t dt + O_t dB_t$.

$$\Rightarrow X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \int_0^t d\langle X, Y\rangle_s$$
$$= X_0 Y_0 + \int_0^t X_s N_s ds + \int_0^t X_s O_s dB_s + \int_0^t Y_s K_s ds + \int_0^t Y_s L_s dB_s + \int_0^t L_s O_s ds$$
$$= X_0 Y_0 + \int_0^t (X_s N_s ds + Y_s K_s + L_s O_s)\, ds + \int_0^t (X_s O_s + Y_s L_s)\, dB_s$$

The Itô martingale representation theorem

Notation
A martingale M_t assuming values in $\mathbb{R}$ w.r.t a Brownian filtration $\mathcal{F}_t$ is called ***Brownian martingale***.

The next theorem will help us to construct the completeness of the market (chapter 2.3) and supports the deduction of portfolio optimization (chapter 3).

1.25 Theorem (The martingale representation theorem)
Let $\{(M_t, \mathcal{F}_t)\}_{t\in[0,T]}$ be a Brownian martingale with $E\left[M_t^2\right] < \infty \; \forall \; t \in [0,T]$, B_t a m-dimensional Brownian motion.
$\Rightarrow \exists$ a $\mathcal{F}_t$-adapted process $\kappa : [0,T] \times \Omega \to \mathbb{R}^m$ with $E\left[\int_0^T \|\kappa(s,\omega)\|^2 ds\right] < \infty$ with

$$M_t(\omega) = M_0(\omega) + \int_0^t \kappa(s,\omega)^t \cdot dB(s,\omega)$$

Proof: see Korn [4], pp. 81-85.
The proof is very technical and basically consists on the comparison of vector spaces and their equality.

Remark

a) κ is $P \otimes \lambda$ unique
b) For a local Brownian martingale theorem 1.25 is still valid (with a matching localization).

1.3 Solutions of stochastic differential equations

For modelling e.g. capital processes in chapter 2.2, we will need stochastic differential equations (= SDEs) and their solutions. The next theorem will help us to solve them.

1.26 Theorem
Let $\{(B_t, \mathcal{F}_t)\}_{t\in I}$ be a m-dimensional Brownian motion, σ_j, $j = 1, \dots, m$, and b $\mathcal{F}_t$-adapted processes with

i) $\sigma_j : [0, T] \times \Omega \to \mathbb{R}$, with $(t, \omega) \mapsto \sigma_j(t, \omega)$

$b : [0, T] \times \Omega \to \mathbb{R}$, with $(t, \omega) \mapsto b(t, \omega)$

ii) $\int_0^t |b(s)|ds < \infty \ \forall\ t \in I$

$\int_0^t \sigma_j(s)^2 ds < \infty \ \forall\ t \in I$ and for $j = 1, \dots, m$

Then the homogeneous SDE

$$\begin{aligned} dP(t) &= P(t)\left(b(t)dt + \sum_{j=1}^{m} \sigma_j(t)dB_j(t)\right) \qquad (HSDE) \\ P(0) &= p \end{aligned}$$

has the unique solution

$$P(t) = p \cdot \exp\left(\int_0^t (b(s) - \frac{1}{2}\sum_{j=1}^{m} \sigma_j(s)^2)ds + \sum_{j=1}^{m} \int_0^t \sigma_j(s)dB_j(s)\right)$$

<u>Proof</u>

i) $P(0) = p \cdot \exp(0) = p$

ii) Let be $m = 1$ (otherwise we apply the multi-dimensional Itô formula, see theorem A.8, Appendix A.2).

$Z_t := 0 + \int_0^t \left(b(s) - \frac{1}{2}\sigma(s)^2\right) ds + \int_0^t \sigma(s)dB_s$
$f(z) := p \cdot \exp(z)$

We apply the Itô formula 1.23:

$$\begin{aligned}
f(Z_t) &= p + \int_0^t \left[f'(Z_s)\left(b(s) - \frac{1}{2}\sigma(s)^2\right) + \frac{1}{2} f''(Z_s)\sigma(s)^2 \right] ds \\
&\quad + \int_0^t f'(Z_s)\sigma(s) dB_s \\
&[f(Z_s) = f'(Z_s) = f''(Z_s) = p \cdot \exp(Z_s) = P(s)] \\
\Rightarrow \quad P(s) &= p + \int_0^t P(s)b(s)ds + \int_0^t P(s)\sigma(s)dB_s \\
\Rightarrow \quad dP(s) &= P(s)\left(b(s)ds + \sigma(s)dB_s\right) \\
P(t) = f(Z_t) &= p \cdot \exp\left(\int_0^t (b(s) - \frac{1}{2}\sigma(s))ds + \int_0^t \sigma(s)dB_s \right)
\end{aligned}$$

For the $P \otimes \lambda$-uniqueness of the solution we define $Z_t := P(t)^{-1}$ and assume that $\widetilde{P(t)}$ is another solution for $(HSDE)$.

$$\begin{aligned}
Z_t = P(t)^{-1} &= p \cdot \exp\left(\int_0^t \left(\frac{1}{2}\sigma(s)^2 - b(s)\right) ds + \int_0^t -\sigma(s)dB_s \right) \\
\Rightarrow dZ_t &= Z_t \left(\left(\frac{1}{2}\sigma(t)^2 - b(t) + \frac{1}{2}\sigma(t)^2 \right) ds - \sigma(t)dB_t \right) \\
&= Z_t \left(\left(\sigma(t)^2 - b(t)\right) ds - \sigma(t)dB_t \right)
\end{aligned}$$

We apply the partial integration, theorem 1.24, and get

$$\begin{aligned}
\widetilde{P(t)}Z_t &= 1 + \int_0^t \widetilde{P(s)}Z_s \left(\left(\sigma(s)^2 - b(s) + b(s) - \sigma(s)^2\right) ds + \left(\sigma(s) - \sigma(s)\right) dB_s \right) \\
\Rightarrow \quad \widetilde{P(t)}Z_t &= 1 \Leftrightarrow \widetilde{P(t)} = \frac{1}{Z_t} = P(t)
\end{aligned}$$

With $p = P(0) = \widetilde{P(0)}$ the solution is unique. ■

But more complicated SDE will occur. For the general case of theorem 1.26 we have the following observation.

1.27 Theorem (Variation of constants)
Assumptions:

- σ_j, b and $\{(B_t, \mathcal{F}_t)\}_{t\in I}$ as in theorem 1.26
- $x \in \mathbb{R}$
- β, S_j $\mathcal{F}_t$-adapted, real valued processes satisfying $\beta, S_j \in \mathcal{C}^0$, $\int_0^t (|\beta(s)|)ds < \infty$ and $\int_0^t (S_j(s))^2 ds < \infty$ P-a.s. $\forall\, t \in I$ and $j = 1, \ldots, m$

Then the SDE

$$dX(t) = (b(t)X(t) + \beta(t))\,dt + \sum_{j=1}^{m} (\sigma_j(t)X(t) + S_j(t))\,dB_j(t) \qquad (SDE)$$
$$X(0) = x$$

has the unique solution w.r.t. $\lambda \otimes P$ $\{(X_t, \mathcal{F}_t)\}_{t\in I}$ with:

$$X(t) = R(t)\left(x + \int_0^t \left(\frac{\beta(s)}{R(s)} - \sum_{j=1}^{m}\sigma_j(s)S_j(s)\right) ds + \sum_{j=1}^{m}\int_0^t \frac{S_j(s)}{R(s)} dB_j(s)\right)$$

in which

$$R(t) = \exp\left(\int_0^t (b(s) - \frac{1}{2}\|\sigma(s)\|^2)ds + \int_0^t \sigma(s)^t dB_s\right)$$

is the unique solution of the HSDE

$$dR(t) = R(t)\left(b(t)dt + \sigma(s)^t dB_t\right)$$
$$R(0) = 0$$

For proof and annotations compare Korn [4], pp. 62-64.
Analogous to theorem 1.26; apply the partial integration for X_t.

Now, we are ready to model stock prices and other significant properties of the market. The presented (especially stochastic) tools form the basis for a mathematical depiction of the financial world.

2 A model of the financial market

We want to describe an economic market in a mathematical way. Therefore, we need to make assumptions and we have to admit restrictions in our model to be able to apply the presented instruments and tools. During the depiction we will collect all these assumptions and restrictions and summarize them at the end of the chapter into an overview. As we mentioned in the introduction, our portfolio only contains stocks and bonds.

2.1 Modelling prices

To get to know the approach of modelling prices we begin with a 'risk-free' asset: the bond. First, what is the meaning of 'risk-free'? 'Risk-free' tells us that the security does not depend on chance or random which means that the future price of a bond is predictable for the investor. By analyzing former performances and future expectations he gets an *expected* yield or an *expected* price for the asset.

The bond price

Let $P_0(t)$ be the price of the bond at time t, $t \in [0, T]$, $T < \infty$. We buy at $t = 0$ and pay $P_0(0) := p_0$. Suppose after one period (for example one year, $t = 1$) the first *expected, constant* interest rate r ($r \in \mathbb{R}^+$) is payed. Our capital rises to $P_0(1) = p_0 + r \cdot p_0 = (1 + r) \cdot p_0$. Assume now that this interest rate is credited n-times a period, then we have (binomial coefficient):

$$
\begin{aligned}
P_0(1) &= \left(p_0 + \frac{r}{n}p_0\right) + \left(p_0 + \frac{r}{n}p_0\right)\left(\frac{r}{n}\right) + \ldots + \left(p_0 + \frac{r}{n}p_0\right)\left(\frac{r}{n}\right)^{n-1} \\
&= p_0\left(1 + \binom{n}{1}\left(\frac{r}{n}\right) + \ldots + \binom{n}{1}\left(\frac{r}{n}\right)^{n-1} + \left(\frac{r}{n}\right)^n\right) \\
&= p_0\left(1 + \frac{r}{n}\right)^n
\end{aligned}
$$

For $n \to \infty$ follows: $P_0(1) = p_0 \cdot \exp(r \cdot 1)$ or rather

$$
P_0(t) = p_0 \cdot \exp(rt) \qquad \text{for } t \in I \tag{2.1}
$$

But what if the interest rate r is not constant? We know that some yields of bonds are floating in dependence of time, but they are no subject of chance. Hence, we write (2.1) for a continuous-time (i.e. non-constant, time dependent and continuous, means $r \in \mathcal{C}^0$) rate of interest:

$$
P_0(t) = p_0 \cdot \exp\left(\int_0^t r(s)ds\right) \qquad \text{for } t \in I \tag{2.2}
$$

We note that (2.2) is the unique solution for the differential equation

$$dP_0(t) = P_0(t)r(t)dt \quad \text{with } P_0(0) = p_0,\ t \in I \qquad (DE)$$

$$\Leftrightarrow \quad P_0(t) = p_0 + \int_0^t P_0(s)r(s)ds$$

With (2.1) or rather (2.2) we have found a mathematical depiction of the bond price.

Stock prices

What are the similarities and differences of a stock price compared to a bond price? The stock price is driven by chance; the investor takes over a certain risk when he buys stocks. This risk has to be compensated with a higher interest rate. This rate can be deducted from the former performances and data of the stock and from the expectations for the asset's future. Thus, this part of the stock price can be determined and is consequently a sort of *expected* price. Concluding, a stock price is similar to the bond one, but cannot be identified in the same manner applied before. Under (2.1) we chose the 'log-linear-approach'.

Assume, n stocks are given. The i-th stock price at time t is $P_i(t)$, $\tilde{b_i}$ the expected rate of return and $P_i(0) = p_i$ the actual price at time $t = 0$. For a bond we have (compare (2.1)):

$$\ln(P_0(t)) = \ln(p_0) + r \cdot t$$

Hence, we derive the following price for stock i:

$$\ln(P_i(t)) = \underbrace{\overbrace{\ln(p_i)}^{\text{price at } t=0} + \overbrace{\tilde{b_i} \cdot t}^{\text{return rate}}}_{\text{expected price}} + \text{'chance'}$$

Let us describe 'chance' (often called 'white-noise' or 'white-noise-effect'). What demands do we have on the random influencing stock prices?

i) It has to be without tendency, mathematically this means $E[\text{'chance'}] = 0$.

ii) It has to be without memory, i.e. 'chance at time t' is independent of 'chance at time s', $s < t$, $s, t \in I$.

iii) It depends on time t.

iv) It describes the sum of the deviation of the real price $P_i(t)$ from the calculated or rather expected price $\ln(p_i) + \tilde{b_i} \cdot t$ on $[0, T]$. If these deviations are independent, we can assume that 'chance' is $N(0, \sigma^2 t)$, $\sigma \in \mathbb{R}^+$, distributed which can be concluded by the central marginal theorem (see Bauer [6] or Michel [10]).

Then, a definition of 'chance' is suggested: $Y(t) := \ln(P_i(t)) - ln(p_i) - \tilde{b_i} \cdot t$

$Y(t)$ fulfils i) and iii). Moreover it shall be that $Y(t) - Y(s)$ depends on $(t - s)$ and is independent of $Y(u)$ for $u \leq s$. This fulfils i),ii) and iv). In other words $Y(t) - Y(s) \sim N(0, \sigma^2(t - s))$.

All these properties of 'chance' are held by the Brownian motion $\{(B_t, \mathcal{F}_t)\}_{t \in I}$ with filtration $\{\mathcal{F}_t\}_{t \in I}$.

But what about the Markowitz' volatility?

The solution for this is obvious. As we deducted the expected interest rate $\tilde{b_i}$, it is possible to get the volatility of a stock i from its former data, variation and future expectation. It is suggested that this can be described by the variance σ_i of the underlying. But we have to pay attention: stocks are correlated among themselves. The volatility σ_{ij} expresses the dependence of asset i from asset j, σ_{ii} is then the calculated variation. All this leads to the following equation:

$$\underbrace{\ln(P_i(t,\omega))}_{\text{actual price}} = \underbrace{\ln(p_i)}_{\text{price at t=0}} + \underbrace{\tilde{b_i} \cdot t}_{\text{expected yield}} + \underbrace{\sigma_{ii} B_t(\omega)}_{\text{volatility connected with 'chance'}}$$

$$\Rightarrow \qquad P_i(t,\omega) = p_i \cdot \exp\left(\tilde{b_i} \cdot t + \sigma_{ii} B_t(\omega)\right) \qquad \forall\ (t,\omega) \in I \times \Omega \tag{2.3}$$

Because (2.3) is only for one stock, we have to modify for the general case to

$$P_i(t,\omega) = p_i \cdot \exp\left(\tilde{b_i} \cdot t + \sum_{j=1}^{m} \sigma_{ij} B_j(t,\omega)\right) \qquad \text{for } i = 1, \ldots, n \tag{2.4}$$

in which $B(t,\omega) = (B_1(t,\omega), \ldots, B_m(t,\omega))^t$ is a m-dimensional Brownian motion.

Remarks

i) In the case $n = m$ σ_{ij} represent the coherence between stock i and j.

ii) We note that $\sum_{j=1}^{m} \sigma_{ij} B_j(t,\omega)$ is $N\left(0, \sum_{j=1}^{m} \sigma_{ij}^2 t\right)$ distributed.

$$\Rightarrow \quad \ln(P_i(t,\omega)) \sim N\left(\ln(p_i) + \tilde{b_i} t, \sum_{j=1}^{m} \sigma_{ij}^2 t\right)$$

iii) Abbreviation: $P_i(t,\omega) := P_i(t)$

We define $b_i := \tilde{b_i} + \frac{1}{2} \sum_{j=1}^{m} \sigma_{ij}(s)^2$, $i = 1, \ldots, n$, and apply theorem 1.26 to (2.4).

We see that

$$P_i(t,\omega) = p_i \cdot \exp\left(\tilde{b_i} t + \sum_{j=1}^{m} \sigma_{ij} B_j(t,\omega)\right) \quad \text{for } i = 1,\dots,n$$

is the unique solution for the HSDE

$$\begin{aligned} dP_i(t) &= P_i(t)\left(b_i dt + \sum_{j=1}^{m} \sigma_{ij} dB_j(t)\right) \\ P_i(0) &= p_i \qquad \text{for } i = 1,\dots,n \end{aligned} \tag{2.5}$$

With (2.5) we have achieved the ***equation of stock prices***, depicted as Itô processes.

2.1 Lemma (Properties of stock prices)

Let $b_i := \tilde{b_i} + \frac{1}{2}\sum_{j=1}^{m} \sigma_{ij}^2$. With (2.4) we have

i) $E[P_i(t)] = p_i \cdot \exp(b_i \cdot t)$

ii) $Var(P_i(t)) = p_i^2 \cdot \exp(2b_i \cdot t) \cdot \left(\exp\left(\sum_{j=1}^{m} \sigma_{ij}^2 t\right) - 1\right)$

iii) $Y_t := a \cdot \exp\left(\sum_{j=1}^{m} \left(c_j B_j(t) - \frac{1}{2}\sigma_{ij}^2 \cdot t\right)\right)$,
$a, c_j \in \mathbb{R}$, $j = 1,\dots,m$ is a martingale with $E[Y_t] = 1$.

Proof: see appendix B.
W.l.o.g. set $m = 1$, use the definition of $P_i(t)$ and the properties of B_t (i.e. $B_t \sim N(0,t)$ and $B_t - B_s$ independent of $\mathcal{F}_s$). Compute the martingale condition $E[Y_t|\mathcal{F}_s] = Y_s$.

Using the definition of b_i we confine that

$$\begin{aligned} P_i(t,\omega) &= p_i \cdot \exp(b_i t) \cdot \underbrace{\exp\left(\sum_{j=1}^{m} \sigma_{ij} B_j(t,\omega) - \frac{1}{2}\sigma_{ij}^2 t\right)}_{\varrho} \\ P_i(0,\omega) &= p_i \qquad i = 1,\dots,n \end{aligned} \tag{2.6}$$

The figure shows four possible price developments of a single stock received by four different paths of the Brownian motion.

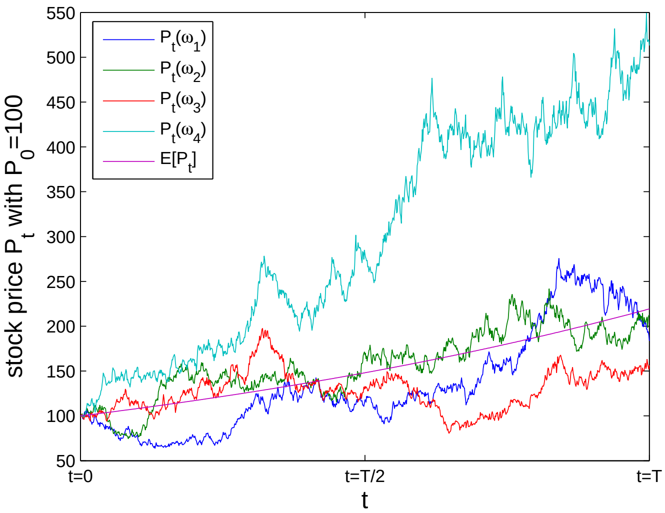

Figure 2.1: Possible stock prices and the stock price expectation

Notation

i) $p_i \cdot \exp(b_i t)$ is the ***expected stock price***,
$b = (b_1, \ldots, b_n)^t$ is the ***vector of expected (anticipated) returns***.

ii) ϱ models the deviation from the expected price ($\rightsquigarrow$ 'chance'),
$\sigma := \begin{pmatrix} \sigma_{11} & \ldots & \sigma_{1m} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \ldots & \sigma_{nm} \end{pmatrix}$ is the ***matrix of volatility***

iii) $P_i(t, \omega) : [0, T] \times \Omega \to \mathbb{R}$ is called ***geometric Brownian motion*** with ***drift*** b_i and ***diffusion*** or ***volatility*** $\sigma_{i\cdot} := (\sigma_{i1}, \ldots, \sigma_{im})$

However, we see that this model is not continuous-time: returns and volatilities do not depend on time which is not realistic in an economic market. If we assume that the expected returns $b_i(t)$ and volatilities $\sigma_{ij}(t)$ are time-dependent, integrable and continuous (i.e. $b, \sigma \in \mathcal{C}^0$), then we get for the bond and stock prices with $b_i(s, \omega) = \tilde{b_i}(s, \omega) + \frac{1}{2} \sum_{j=1}^{m} \sigma_{ij}^2(s, \omega)$:

$$\begin{aligned} P_0(t, \omega) &= p_0 \cdot \exp\left(\int_0^t r(s, \omega) ds \right) \\ P_0(0, \omega) &= p_0 \\ P_i(t, \omega) &= p_i \cdot \exp\left(\int_0^t \tilde{b_i}(s, \omega) ds + \int_0^t \sum_{j=1}^{m} \sigma_{ij}(s, \omega) dB_j(s, \omega) \right) \\ P_i(0, \omega) &= p_i \qquad i = 1, \ldots, n \text{ and } \forall\, t \in I \end{aligned} \tag{2.7}$$

Demands on our price model

- $r(t) = r(t,\omega)$, $b(t) = b(t,\omega) = \begin{pmatrix} b_1(t,\omega) \\ \vdots \\ b_n(t,\omega) \end{pmatrix}$,
  $\sigma(t) = \sigma(t,\omega) = \begin{pmatrix} \sigma_{11}(t,\omega) & \dots & \sigma_{1m}(t,\omega) \\ \vdots & \ddots & \vdots \\ \sigma_{n1}(t,\omega) & \dots & \sigma_{nm}(t,\omega) \end{pmatrix}$
  are $\mathcal{F}_t$-adapted and regular bounded processes, r, b and $\sigma \in \mathcal{C}^0$.
- $\sigma(t)\sigma(t)^t$: $\exists\, K > 0$ with $x^t\sigma(t)\sigma(t)^t x \geq K x^t x \quad \forall\, x \in \mathbb{R}$ and $\forall\, t \in I$

  which means that σ is uniformly positive definite.

With this we apply theorem 1.26, chapter 1.3, and write (2.7) as Itô processes.

$$\begin{aligned} dP_0(t) &= P_0(t)r(t)dt \\ P_0(0) &= p_0 \\ dP_i(t) &= P_i(t)\left(b_i(t)dt + \sum_{j=1}^{m}\sigma_{ij}(t)dB_j(t)\right) \\ P_i(0) &= p_i \quad\quad i = 1,\dots,n \end{aligned} \tag{2.8}$$

Remarks

- The bond is not 'risk-free' anymore, but its dependence on 'chance' is severely restricted because $r(t,\omega)$ is regular bounded. In practice this comes closer to reality as the suggestion that the bond is a complete 'risk-free' asset.
- $(t,\omega) \in [0,T] \times \Omega$ describes the market situation.

2.2 Portfolio processes

What kind of possibilities does an investor have to act or to react on the market? First, he has a certain amount of money which he wants to invest (starting capital). Then, after his first investment, he can redistribute his capital. This means to sell assets and to invest in other securities. On the other hand, he can consume parts of his property by selling without reinvestment. A negative consumption would be, if the investor added money to his existing portfolio. We will not regard this case. Besides the actions of the investor, we have to formulate master conditions for our market. Some are obvious:

- The investor may not have insider information.
- The investor is not able to influence the price trend of assets (maxim of the 'small investor')

- At the beginning ($t = 0$) the investor has a certain starting capital ($x > 0$, initial value). This starting capital has to be invested (normally completely) in n stocks and one bond.

And some are necessary for our theory:

- Portfolios are *self-financing*, i.e. every change concerning the investor's capital results from consumption or redistribution.
- Assets are divisible by any number.
- There are no costs of transaction.
- Short sellings and credits are allowed:
  A negative share of a bond symbolizes a credit. The interest rate r is for an investment in a bond as well as for a credit (compare the next point). A negative share of a stock means the investor owes to another investor these stocks (short sellings).
- The market is complete, every investor gets the same conditions. The information about securities are obtainable for everyone. Interest rate r for credits and bonds is equal for every participant (see previous aspect). This implies that $r(t, \omega)$ only depends on the market situation $(t, \omega) \in [0, T] \times \Omega$ and not on the credit standing status of the investor. Market imperfections are excepted.

An introduction to the expression 'self-financing'

Let us develop our theory by regarding an example. Let $x \in \mathbb{R}^+$ be the investor's *starting capital* to be invested for two periods of time, i.e. $t \in \{0, 1, 2\}$. The *capital process* $X(t)$ describes the investor's wealth at time t. His capital rises in the case of win and it descends in the case of loss or consumption which is understood as the *consumption process* $C(t) \geq 0$ with $C(0) = 0$.

Suppose we only trade one bond and one stock. Their prices are defined as $P_0(t)$ and $P_1(t)$, see chapter 2.1. At time $t = 0$ the investor determines the shares of the starting capital x to be put into the bond ($:= \vartheta_0(0)$) and into the stock ($:= \vartheta_1(0)$). This leads to the following approach:

$$X(0) = \vartheta_0(0) \cdot P_0(0) + \vartheta_1(0) \cdot P_1(0) = x \tag{1}$$

After the first period ($t = 1$), if the investor consumes parts of his wealth, will say $C(1) > 0$, we have

$$X(1) = \vartheta_0(0) \cdot P_0(1) + \vartheta_1(0) \cdot P_1(1) - C(1) \tag{2}$$

In addition to that he can redistribute his portfolio which means:

$$X(1) = \vartheta_0(1) \cdot P_0(1) + \vartheta_1(1) \cdot P_1(1) \tag{3}$$

We regard this in an abstract way. What is $X(1)$ from a general point of view?

$X(1)$ = *starting capital* + *win/loss bond* + *win/loss stock* - *consumption*

Nevertheless, it is also, see equation (3):

$X(1) = \textit{share bond}_{t=1} \times \textit{bondprice}_{t=1} + \textit{share stock}_{t=1} \times \textit{stockprice}_{t=1}$

From equations (1) - (3) we obtain:

$$X(1) = x + \vartheta_0(0)(P_0(1) - P_0(0)) + \vartheta_1(0)(P_1(1) - P_1(0)) - C(1) \quad (4)$$

$$X(1) = \vartheta_0(1) \cdot P_0(1) + \vartheta_1(1) \cdot P_1(1) \quad (5)$$

Analogous we get for $t = 2$:

$$X(2) = x + \sum_{i=1}^{2} \vartheta_0(i-1)(P_0(i) - P_0(i-1)) + \sum_{i=1}^{2} \vartheta_1(i-1)(P_1(i) - P_1(i-1)) - \sum_{i=1}^{2} C(i) \quad (6)$$

$$\Leftrightarrow \quad X(2) = \vartheta_0(2) \cdot P_0(2) + \vartheta_1(2) \cdot P_1(2) \quad (7)$$

The fact that equations (6) and (7) or rather (4) and (5) are equal is called *'self-financing'*.

This makes sense for a discrete-time model, but we need a continuous-time one which requires that the investor may trade or consume at every time $t \in I$. However, this means that the occurring sums in (6) or (4) merge into integrals (the differences are getting infinitesimal small). The result is

$$X(t) = x + \int_0^t \vartheta_0(s) dP_0(s) + \int_0^t \vartheta_1(s) dP_1(s) - \int_0^t C(s) ds \quad (2.9)$$

We recall that P_i, $i = 1, 2$, are Itô processes and we have to make restrictions such that the integrals in (2.9) exist. With this in mind we specify as follows:

2.2 Definition

a) A ***trading strategy*** is a $\mathcal{F}_t$-adapted process $\vartheta : [0, T] \times \Omega \to \mathbb{R}^{n+1}$, $\vartheta(t) := \vartheta(t, \omega) := (\vartheta_0(t, \omega), \ldots, \vartheta_n(t, \omega))^t$ such that

   i) $\int_0^T |\vartheta_0(s)| ds < \infty$ P-a.s. and

   ii) $\int_0^T (\vartheta_i(s) P_i(s))^2 ds < \infty$ P-a.s. for $i = 1, \ldots, n$.

b) $x := \sum_{i=0}^{n} \vartheta_i(0) \cdot p_i$ is called ***initial value*** of ϑ.

c) Let $C : [0, T] \times \Omega \to \mathbb{R}_0^+$, $C(t) := C(t, \omega) \geq 0$ be a $\mathcal{F}_t$-adapted process with $\int_0^T C(s) ds < \infty$ P-a.s.. Then $C(t)$ is a ***consumption process***.

Annotations

- That a trading strategy is $\mathcal{F}_t$-adapted is the conversion of the demand that the investor does not have any insider information. His decision to sell or to buy only depends on the market situation $(t, \omega) \in [0, T] \times \Omega$.
- The initial value x is (normally) completely invested in the portfolio.
- $\vartheta \in \mathbb{R}^{n+1}$ is held, because in the cases of credits and short sellings its value can be negative.
- We remark that with $C(t) \geq 0 \ \forall \ t \in I$ no added money is admitted.

2.3 Definition
Let $x \in \mathbb{R}^+$ be an initial value and ϑ a trading strategy.

a) Then $X : [0, T] \times \Omega \to \mathbb{R}$,
$X_t := X(t) := X(t, \omega) := \sum_{i=0}^{n} \vartheta_i(t, \omega) \cdot P_i(t, \omega) := \sum_{i=0}^{n} \vartheta_i(t) \cdot P_i(t)$
is the ***wealth*** or ***capital process*** w.r.t ϑ.

b) For $X_t > 0$ P-a.s.
$\pi : [0, T] \times \Omega \to \mathbb{R}^n$, $\pi(t) := \pi(t, \omega) := (\pi_1(t, \omega), \ldots, \pi_n(t, \omega))^t$ with

$$\pi_i(t, \omega) := \frac{\vartheta_i(t, \omega) P_i(t, \omega)}{X(t, \omega)} \quad i = 1, \ldots, n$$

is called ***portfolio process*** w.r.t. ϑ.

The capital process X_t allows to determine directly the actual value of the investor's portfolio.

As we introduced the term *'self-financing'*, we came to the conclusion that

'actual wealth' = 'initial value' + 'win/loss assets' - *'consumption'*

must be valid. We formalize this in the following definition.

2.4 Definition
Let ϑ be a trading strategy , C a consumption process and X the accompanying capital process.

a) Then (ϑ, C) is ***self-financing***, if the following equation is held P-a.s. $\forall \ t \in I$:

$$X(t) = x + \sum_{i=0}^{n} \int_0^t \vartheta_i(s) dP_i(s) - \int_0^t C(s) ds \tag{2.10}$$

We remember (2.8) and realize that

$$\begin{aligned} X(t) = x &+ \int_0^t \vartheta_0(s) P_0(s) r(s) ds + \sum_{i=1}^{n} \int_0^t \vartheta_i(s) P_i(s) b_i(s) ds \\ &+ \sum_{i=1}^{n} \int_0^t \sum_{j=1}^{m} \vartheta_i(s) P_i(s) \sigma_{ij}(s) dB_j(s) - \int_0^t C(s) ds \end{aligned} \tag{2.11}$$

b) If (ϑ, C) is self-financing and $X(t) \geq 0$ P-a.s. $\forall\, t \in I$,
a portfolio process π is called ***self-financing portfolio process*** (π, C) w.r.t. (ϑ, C).

Remarks

- $X(t)$ and all integrals in (2.10) and (2.11) are defined and exist, because of the assumptions for C, ϑ_i, P_i, $i = 0, \ldots, n$, in definition 2.2 and for r, b, and σ in chapter 2.1 .
- If $X(t)$ and $P_i(t)$, $i = 1, \ldots, n$, are known, we have $(\vartheta, C) \Leftrightarrow (\pi, C)$.
- Furthermore it is

$$(1 - \pi^t \cdot e^n) = \frac{X - \sum\limits_{i=1}^{n} \vartheta_i \cdot P_i}{X} = \frac{\vartheta_0 \cdot P_0}{X}$$

To get $X(t)$ in form of a SDE, we compute (compare (2.10) and (2.11)):

$$\begin{aligned}
dX_t &= \sum_i \vartheta_i dP_i - Cdt \\
&= \vartheta_0 P_0 r dt + \sum_i \vartheta_i P_i b_i dt + \sum_i \sum_j \vartheta_i P_i \sigma_{ij} dB_j - Cdt \\
&= (1 - \pi^t e^n) X_t r dt + \sum_i \pi_i X_t b_i dt + \sum_i \pi_i X_t \sum_j \sigma_{ij} dB_j - Cdt \\
&= (1 - \pi^t e^n) X_t r dt + X_t \pi^t b dt + \underbrace{\sum_i X_t \pi_i \sigma_{i\cdot} dB}_{= X_t \pi^t \sigma dB} - Cdt \\
&= X_t \pi^t [(b - e^n r) dt + \sigma dB] + (X_t r - C) dt
\end{aligned}$$

Notation

The SDE

$$\begin{aligned}
dX_t &= (X_t r(t) - C(t))dt + X_t \pi(t)^t [(b(t) - r(t) e^n) dt + \sigma(t) dB_t] \\
X_0 &= x
\end{aligned} \qquad (CE)$$

is called ***capital equation (CE)***.

We see that $r(t)$, $b(t)$, $\sigma(t)$ and $C(t)$ fulfil the assumptions of theorem 1.27 (variation of constants). To get an unique solution for (CE), it is required that

$$\int_0^T \pi_i(t)^2 dt < \infty \qquad P\text{-a.s., } i = 1, \ldots, n \qquad (A)$$

Notation (compare with definition 2.4 b)

If (CE) has an unique solution $X_t := X^{\pi,C}(t,\omega) := X^{\pi,C}(t)$ with

$$\int_0^T (X_t \pi_i(t))^2 dt < \infty \qquad P\text{-a.s., } i = 1, \dots, n \tag{B}$$

$\pi(t)$ is called ***self-financing portfolio process*** and we denote X_t as ***accompanying*** or ***corresponding capital process***.

Let us examine the restrictions (A) and (B) and ii) in definition 2.2. First, we remark that

$$\int_0^T (X_t \pi_i(t))^2 dt < \infty \Leftrightarrow \int_0^T (\vartheta_i(t) P_i(t))^2 dt < \infty$$

Then, we know that X_t is continuous. This implies that with $\int_0^T \pi_i(t)^2 dt < \infty$ also $\int_0^T (X_t \pi_i(t))^2 < \infty$ is valid (in other words: $(A) \Rightarrow (B)$).

If we assume (A), our (CE) is solved by a capital process X_t with $X_t > 0$. This results from the explicit solution for (CE) (compare theorem 1.27). Whereas under restriction (B) for (CE) the solution X_t is in $\mathbb{R}$, i.e. the investor could go bankrupt ($X_t = 0$) or get into a 'debt position' ($X_t < 0$).

2.5 Definition
Let (ϑ, C) / (π, C) be a self-financing trading strategy ϑ / portfolio process π with a consumption process C and $x > 0$ as initial value. If the capital process X_t holds $X_t \geq 0$ P-a.s. $\forall\, t \in I$, we call X_t ***admissible*** with initial value x. Further, we define

$$D_\vartheta(x) := \{(\vartheta, C) | (\vartheta, C) \text{ is self-financing with admissible } X_t \geq 0, X(0) = x\}$$
$$D_\pi(x) := \{(\pi, C) | (\pi, C) \text{ is self-financing with admissible } X_t \geq 0, X(0) = x\}$$

2.6 Example
Let $x > 0$ be the initial value. Suppose the investor does not want to trade, which means that the shares of securities stay constant (i.e. $\vartheta(t) \equiv \vartheta$). Moreover, he does not consume any of his capital during the investment, so it is $C(t) \equiv 0$. Then, we have with $\pi(t) = \vartheta \cdot \frac{P(t)}{X(t)}$ and $X(0) = x$:

$$\begin{aligned} & dX(t) = X(t) r(t) + X(t) \pi(t)^t \left[(b(t) - r(t) e^n) \, dt + \sigma(t) dB_t \right] \\ \Rightarrow \quad & dX(t) = X(t) \underbrace{\left[r(t) + \pi(t)^t b(t) - \pi(t)^t r(t) e^n \right]}_{\simeq b(t)} dt + X(t) \underbrace{\pi(t)^t \sigma(t)}_{\simeq \sigma(t)} dB_t \end{aligned}$$

With theorem 1.26 we get as solution:

$$
\begin{aligned}
X(t) =& x \cdot P(t) \\
P(t) =& \exp\left(\int_0^t r(s) + \pi(s)^t (b(s) - r(s)e^n) - \frac{1}{2}\|\pi(s)^t\sigma(s)\|^2\right) ds \\
& \cdot \exp\left(\int_0^t \pi(s)^t \sigma(s) dB_s\right)
\end{aligned}
$$

On implication is surely that $X_t \geq 0 \ \forall \ t \in I$
$\Rightarrow (\vartheta(t), C(t)) = (\vartheta, 0) \in D_\vartheta$ or $(\pi(t), C(t)) = (\pi(t), 0) \in D_\pi$.

2.3 The complete market

In the previous section we introduced the term 'complete market' in an economical sense. In this part we deal with the completeness of the market in a mathematical way.

Normally, the investor wants to maximize his yields or he wants to live according to a consumption process determined by him in advance. He may also aim for certain predestined terminal value in the end of the investment.

Thus, we have to find a *discount factor*, especially for the last two aspects. It must allow to calculate back from the terminal aims or occurrences to any time $t \in [0, T]$ of the beginning investment. As the market is complete in an economical way, we conclude that one part of the discount factor must be the interest rate r of the bond or credits. Hence, we have

$$
\delta_1(t) := \delta_1(t, \omega) := \exp\left(-\int_0^t r(s, \omega) ds\right) = \exp\left(-\int_0^t r(s) ds\right)
$$

Further, we define

$$
\begin{aligned}
\xi(t) := \xi(t, \omega) &:= \sigma(t, \omega)^{-1}(b(t, \omega) - r(t, \omega)e^n) \\
&= \sigma(t)^{-1}(b(t) - r(t)e^n) \\
\delta_2(t) := \delta_2(t, \omega) &:= \exp\left(-\int_0^t \xi(s, \omega)^t dB_s(\omega) - \frac{1}{2}\int_0^t \|\xi(s, \omega)\|^2 ds\right) \\
&= \exp\left(-\int_0^t \xi(s)^t dB_s - \frac{1}{2}\int_0^t \|\xi(s)\|^2 ds\right) \\
\delta(t) &:= \delta_1(t) \cdot \delta_2(t)
\end{aligned}
$$

The next figure shows four possible paths of $\delta(t, \omega)$ with constant market coefficients σ, b and $r \in \mathbb{R}$.

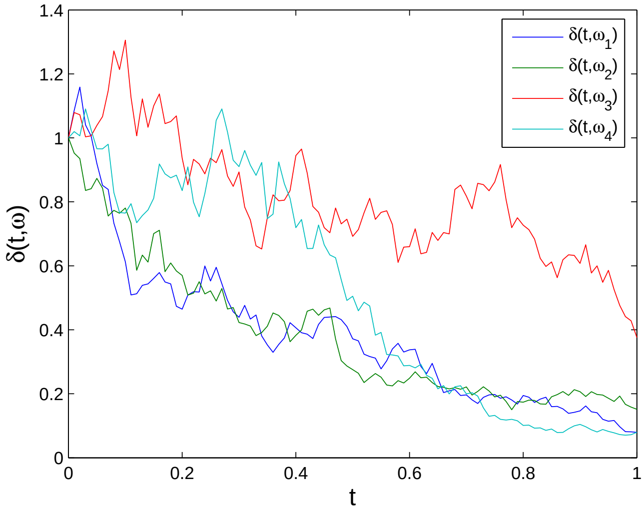

Figure 2.2: The discount factor $\delta(t,\omega)$

Properties of the discount factor δ

First, we remark that $\delta(t) > 0 \; \forall \; t \in I$. As r, b and σ are continuous, regularly bounded and $\mathcal{F}_t$-adapted, $\delta(t)$ is continuous and $\mathcal{F}_t$-adapted as well. Moreover, we see that $\delta(t)$ has the same structure as the unique solution $P(t)$ in theorem 1.26. Hence, we can describe $\delta(t)$ as a SDE or as Itô process:

$$\begin{aligned} d\delta(t) &= -\delta(t)\left(r(t)dt + \xi(t)^t dB_t\right) \\ \delta(0) &= 1 \end{aligned} \tag{2.12}$$

Explanation

- $\xi(t)$ is a *relative market risk premium*. For the better understanding we regard the case $n = 1$ and constant processes r, b and σ. This leads to $\xi \equiv \frac{b-r}{\sigma}$. We recognize this expression: it is the *Sharpe ratio* used for the capital market line (CML, deduced by Tobin from Markowitz' theory). Obviously, ξ indicates the premium for the risk to invest into stocks instead of investing into the bond. Compared to the modelling of stock prices it symbolizes 'chance' for the discount factor.

- For $\delta(t)$ or rather $\delta_2(t)$ we remember the deduction of the stock prices. In equation (2.6) we modelled stock prices and annotated that the expression 'ϱ' stands for the deviation of the expected price. As $\delta_1(t)$ represents a sort of *expected discount factor*, we add a discount factor depending on chance ($= \delta_2(t)$). The solution of 2.12 explains the special form of δ_2 with the additional term $-\frac{1}{2}\int_0^t \|\xi(s)\|^2 ds$.

- Thus, $\delta(t)$ can be regarded as a discount factor depending on the market situation $(t,\omega) \in [0,T] \times \Omega$. $\delta(s,\overline{\omega})$ describes the actual value of money means payed at time s in the market condition $\overline{\omega}$.

Now, with help of $\delta(t)$, we are able to present an important theorem which shows an astonishing feature of the just developed market.

2.7 Theorem (Completeness of the market)
Let be

- π a portfolio process
- C a consumption process
- $x \in \mathbb{R}^+$ the initial value
- $(\pi, C) \in D_\pi(x)$ and
- $\delta(t)$ the above-mentioned discount factor

i) For the accompanying capital process X_t we have:

$$E\left[\delta(t)X_t + \int_0^t \delta(s)C(s)ds\right] \leq x \qquad \forall\ t \in I$$

ii) Let Q be a $\mathcal{F}_T$-measurable random variable, $Q \geq 0$, and let the initial value X_0 hold: $x := E\left[\delta(T)Q + \int_0^T \delta(s)C(s)ds\right]$
$\Rightarrow \exists$ a portfolio process $\pi(t)$, $t \in I$, with $\pi \in D_\pi(x)$ such that the corresponding capital process satisfies

$$X_T = Q \qquad P\text{-a.s.}$$

Proof: see appendix B.
Show that $\delta(t)$, $X_t \in \mathcal{J}$, apply theorem 1.24 to $\delta(t) \cdot X_t$. Define
$X_t := \delta(t)^{-1}E\left[\int_t^T \delta(s)C(s)ds + \delta(T)Q|\mathcal{F}_t\right]$,
$M_t := \delta(t)^{-1}E\left[\int_0^T \delta(s)C(s)ds + \delta(T)Q|\mathcal{F}_t\right]$
and show that $\delta(t)X_t + \int_0^t \delta(s)C(s)ds = M_t$. Furthermore, check that M_t is a martingale, apply theorem 1.25, determine κ and show that $(\pi, C) \in D_\pi(x)$.

To be mentioned: π is the unique portfolio process except from $P \otimes \lambda$ - equivalence.

Interpretation of theorem 2.7

First, we separate

$$E\left[\delta(t)X_t + \int_0^t \delta(s)C(s)ds\right] \text{ into}$$
$$(\star)\ E\left[\delta(t)X_t\right] \text{ and}$$
$$(*)\ E\left[\int_0^t \delta(s)C(s)ds\right].$$

If we interpret $\delta(t)$ as a suitable discount factor, we see that $(\star)$ indicates the expected, discounted capital at time t, whereas $(*)$ stands for the expected, discounted and average consumption. For $t = T$, X_T symbolizes the *terminal* capital. Thus,

$$\underbrace{E\left[\delta(T)X_T\right]}_{\text{discounted, terminal capital}} + \underbrace{E\left[\int_0^T \delta(s)C(s)ds\right]}_{\text{discounted consumption}}$$

manifests the needed initial value to achieve desired aims of the investor, e.g. a given terminal capital or living according to certain consumption. Therefore, part i) restricts the investor's wishful thinking concerning consumption and terminal wealth.

If we regard goals permitted by part i), part ii) tells us that they can be realized with help of these initial endowments. In other words, if we predetermine a consumption process and a terminal wealth satisfying the assumptions

$$E\left[\delta(t)X_t + \int_0^t \delta(s)C(s)ds\right] \leq x \text{ and}$$
$$E\left[\delta(T)Q + \int_0^T \delta(s)C(s)ds\right] = x$$

we can find a portfolio process with $(\pi, C) \in D_\pi(x)$.

Hence, there is one remarkable feature of this theorem:
We can achieve *every* desired terminal capital with help of a suitable, self-financing portfolio process, if we just have *enough starting capital.*

A market with this property is called ***complete market***.

2.8 Example

$X_t := \delta(t)^{-1}$ corresponds to the strategy $(\pi(t), C(t)) := ((\sigma(t)^{-1})^t \xi(t), 0)$ with $X_0 = 1$.

Proof

1. $X_0 = \delta(0)^{-1} = 1$
2. $\delta(t)^{-1} = \exp\left(\int\limits_0^t r(s) + \frac{1}{2}\|\xi(s)\|^2 ds + \int\limits_0^t \xi(s)^t dB_s\right) \qquad (\diamond)$
3. We remember example 2.6 ($C(t) \equiv 0$) and use $X_t = \delta(t)^{-1}$ or rather $x = 1$ and $\pi(t) = (\sigma(t)^{-1})^t \xi(t)$:

$$\begin{aligned}
X_t &= \exp\left(\int\limits_0^t \big(r(s) + ((\sigma(s)^{-1})^t \xi(s))^t (b(s) - r(s)e^n)\big)\, ds\right) \\
&\quad \cdot \exp\left(-\int\limits_0^t \frac{1}{2} \left\| \big((\sigma(s)^{-1})^t \xi(s)\big)^t \sigma(s) \right\|^2 ds\right) \\
&\quad \cdot \exp\left(\int\limits_0^t \big((\sigma(s)^{-1})^t \xi(s)\big)^t \sigma(s) dB_s\right) \\
&= \exp\left(\int\limits_0^t \left(r(s) + \underbrace{\xi(s)^t \overbrace{\sigma(s)^{-1}(b(s) - r(s)e^n)}^{=\xi(s)}}_{=\|\xi(s)\|^2} \right)\right) \\
&\quad \cdot \exp\left(-\int\limits_0^t \frac{1}{2} \left\| \xi(s)^t \sigma(s)^{-1} \sigma(s) \right\|^2 ds\right) \\
&\quad \cdot \exp\left(\int\limits_0^t \xi(s) \sigma(s)^{-1} \sigma(s) dB_s\right) \\
&= \exp\left(\int\limits_0^t (r(s) + \frac{1}{2}\|\xi(s)\|^2) ds + \int\limits_0^t \xi(s)^t dB_s\right) \qquad (\bullet)
\end{aligned}$$

We see $(\diamond) = (\bullet)$. ∎

2.4 Assumptions for the complete market

A small overview concerning the assumptions required for our financial market model.

Economic premises

- n stocks and one bond (no other securities)
- maxim of the 'small investor'
- no insider knowledge
- short sellings and credits possible
- no transaction costs
- no market imperfections, a complete market in the economical way

Mathematical requirements

- self-financing portfolio processes π
- admissible capital processes X_t ($X_t \geq 0$ P-a.s. $\forall\, t \in I$, $X_0 = x > 0$)
- a complete market in the mathematical way
- assets divisible by any number
- restrictions for r, b, σ and the consumption process C:
  - $r(t,\omega)$ $\mathcal{F}_t$-adapted, $\mathbb{R}$-valued, regularly bounded process, $r \in \mathcal{C}^0$
  - $b(t,\omega)$ $\mathcal{F}_t$-adapted, $\mathbb{R}^n$-valued, regularly bounded process, $b \in \mathcal{C}^0$
  - $\sigma(t,\omega)$ $\mathcal{F}_t$-adapted, $\mathbb{R}^{n\times m}$-valued, regularly bounded process, $\sigma \in \mathcal{C}^0$, uniformly positive definite
  - $C(t,\omega)$ $\mathcal{F}_t$-adapted, $\mathbb{R}_0^n$-valued process with $\int C(t)dt < \infty$
- $I = [0,T]$, $T < \infty$, finite time horizon
- $B(t,\omega)$ is a m-dimensional Brownian motion, $\mathcal{F}_t$ the Brownian filtration
- $(t,\omega) \in [0,T] \times \Omega$ describes the market situation at time t
- $(\Omega, \mathcal{F}, P)$ is a complete probability space
- $n = m$, the dimension of the Brownian motion corresponds to the number of stocks

3 Portfolio optimization

In the previous chapter we presented a model of an economical and mathematical complete market. It has a fixed time horizon with $I = [0, T]$ and is continuous in time (i.e. trading and consumption is allowed at any time $t \in I$). Moreover, it comprises $n + 1$ securities, one bond and n stocks. Remember the price description (2.7) driven by a n-dimensional ($n = m$) Brownian motion and influenced by the rates of return $r(t)$ for the bond (possibly randomly), $b_i(t)$, $i = 1, \ldots, n$, for the stocks (randomly fluctuating) and by the volatility coefficients $\sigma_{ij}(t)$, $i, j = 1, \ldots, n$ with its assumptions (see chapter 2.4) and the resulting settings and properties.

The complete market model will allow us to develop a method to optimize our portfolio or respectively portfolio processes: the martingale approach.

The martingale approach was introduced in the 1980ies in different versions by Cox and Huang, Pliska, Karatzas et al, compare p.e. Karatzas, Lehoczky and Shreve [11].

The other main approach to solve the later presented unconstrained portfolio problem (P) was developed by Merton in 1969: the 'stochastic control approach'. As Merton was one of the first introducing and solving the problem, (P) is sometimes called 'Merton's problem'.

The basis for the stochastic control approach is the stochastic control theory. Here, a solution for (P) is computed by solving a Hamilton-Jacobi-Bellmann (HJB) equation in two steps: First, an optimal portfolio and consumption processes are searched in dependence of the unknown optimal expected utility. Putting this solution into the (HJB) equation, the result is a non-linear partial differential equation. It can be solved under special conditions while it is hard to get a solution in the general case. In addition to that the resulting numerical problem often has no explicit solution. For more information refer to Merton [12, 13].

However, we direct our focus to the martingale method.

3.1 The continuous-time portfolio problem

To get to our portfolio problem we take a closer look at the investor's problem: He is endowed with an initial value x. Now, he must choose strategies for his investment and his consumption in dependence of the stocks' price trends. This means he has to determine

- *what* share of *which* stock at *which* time he holds to get a *terminal value* as high as possible and
- *how* much capital he *consumes* at *which* time.

In other words he wants to maximize the utility of terminal wealth and consumption. Thus, we need to find a measure for 'utility' - *the utility function.*

Let us consider the problem from the economic point of view:
First, it is obvious that with additional wealth or consumption the investor has more utility. Mathematically, this implies that the utility function must be strictly increasing.

Further, it would be logical (in an economic sense) that the *additional* profit decreases the higher the unit of utility (i.e. consumption/wealth $\triangleq x$) is.

Viceversa, the loss of profit increases the lower the unit of utility is.
In economic terms: the marginal profit in $x = 0$ is infinite ('a little bit is better than nothing'), whereas it equals zero in $x = \infty$ (saturation effect).

Hence, we denote as follows:

3.1 Definition

i) Let $u : (0, \infty) \to \mathbb{R}$ be a strictly concave function, $u \in \mathcal{C}^1$, such that

   (a) $u'(0) := \lim_{x \downarrow 0} u'(x) = \infty$

   (b) $u'(\infty) := \lim_{x \to \infty} u'(x) = 0$

   Then, u is a ***utility function***.

ii) A function $U : [0, T] \times (0, \infty) \to \mathbb{R}$, $U \in \mathcal{C}^{0,1}$ is called ***utility function*** in the sense of i), if U satisfies

   (a) $U_t(\cdot)$ is utility function for all $t \in I$ and

   (b) $U.(x) \in \mathcal{C}^0$ for fixed $x \in \mathbb{R}^+$

Remarks

- For a detailed description of properties of utility functions read appendix A.3.
- With i) (a) and (b) the requirements for the marginal profit are satisfied ('a little bit is better than nothing' and the saturation effect).

3.2 Example

Examples for utility functions are

- $u(x) = \ln(x)$
- $u(x) = \sqrt{x}$
- $U_t(x) = \exp(-\alpha t) \cdot u(x)$, $\alpha > 0$,
  for a utility function u in the sense of definition 3.1 i).
- $U_t(x) = \exp(-t) \cdot \ln(x)$

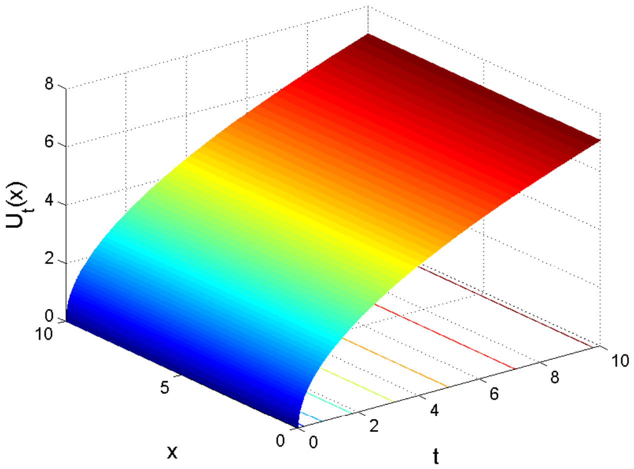

Figure 3.1: Utility function $U_t(x) = \exp(-0.05t)\sqrt{x}$

We pointed out that the investor would like to maximize the utility of the terminal value and consumption, in formula this means:

$$\underbrace{\int_0^T U_1^s(C(s))ds}_{\text{utility of consumption}} \quad + \quad \underbrace{U_2(X_T)}_{\text{utility of terminal value}} \quad \longrightarrow \max!$$

in which U_1, U_2 are utility functions in sense of definition 3.1.

However, we have to consider that we deal with random variables, that is why we use

$$E\left[\int_0^T U_1^s(C(s))ds + U_2(X_T)\right] \longrightarrow \max!$$

We know that X_t has a consumption process $C(t)$ and a portfolio process $\pi(t)$. And as we are in the complete market settings, it yields $(\pi, C) := (\pi(t), C(t)) \in D_\pi(x)$ with x as initial value. Then, we get as provisional result

$$(\bar{P}) \begin{cases} \max\ \Psi(x, \pi, C) = E\left[\int_0^T U_1^s(C(s))ds + U_2\left(X^{x,\pi,C}(T)\right)\right] \\ \text{s.t. } (\pi, C) \in D_\pi(x) \end{cases}$$

Normally, we would restrict the problem $(\bar{P})$ s.t. the occurring expectation is finite. But: the investor's aim is to gain as much (utility) as he can. Thus, a strategy with infinite utility would be the most desirable one. We would exclude an optimal strategy. Consequently, we define:

$$D_\pi^*(x) := \left\{(\pi, C) \in D_\pi(x) | E\left[\int_0^T \left(U_1^s(C(s))\right)^- ds + \left(U_2\left(X^{x,\pi,C}(T)\right)\right)^-\right] < \infty\right\}$$

We see

- a negative, infinite utility is excluded whereas a positive, infinite utility is allowed
- for $U_1 \geq 0, U_2 \geq 0$ we have $D^*_\pi(x) = D_\pi(x)$, otherwise $D^*_\pi(x) \subseteq D_\pi(x)$

With this we obtain the continuous-time problem for the initial value $x \in \mathbb{R}^+$:

$$(P) \begin{cases} \max \ \Psi(x,\pi,C) = E\left[\int_0^T U_1^s(C(s))ds + U_2\left(X^{x,\pi,C}(T)\right)\right] \\ \text{s.t. } (\pi, C) \in D^*_\pi(x) \end{cases}$$

3.3 Definition (Portfolio problem)
The optimization problem (P) is called ***(unconstrained) portfolio problem*** or ***Merton's problem***.

Over the years one has defined many sorts of portfolio problems. Markowitz began with the mean-variance-approach, others admit transaction costs or allow an infinite time horizon, for example. In this thesis the depiction and solution of (P) are severely depending on the complete market settings.

3.2 The martingale approach

The main idea of the martingale approach is a decomposition of the dynamic portfolio problem (P) into a static optimization problem (S) and a representation problem (R).

In the static problem (S) we determine an optimal consumption and terminal wealth each geared to the needs or wishes of the investor. Then, in the representation problem (R), we compute a portfolio process π and a consumption process C which coincide with the solution of (S).

Significant for this approach is the fact that we *already* know the optimal solution for (P) (step 1: solution of (S)), before we have found its representation in form of processes (step 2: solution of (R)).

Advantages and disadvantages

A negative aspect is surely that the martingale approach is strongly restricted to the completeness of the market, mathematically and economically (see chapter 2.4). Some of these assumptions are not really realistic (e.g. market imperfections), but the complete market settings form the basis for the martingale approach. On the other hand this leads to several advantages:

+ non-constant, time- and random-dependent market coefficients
+ the preceding point implies a continuous-time market model, in particular this allows a continuously (negative) investment and consumption at any time $t \in I$.

+ general utility functions: the investor can freely choose the utility functions w.r.t. its definition 3.1. The choice of U_1 and U_2 reflects the investor's investment behaviour: he is able to control which part (terminal wealth or consumption) he emphasizes or neglects.

+ If the solution of (P) has a non-closed form, the 'Monte-Carlo-simulation' as numerical computation often delivers an explicit solution. We will not discuss this case.

(+) For some problems it might be helpful to have an infinite time horizon $I = [0, \infty]$, but in most cases a finite time period $I = [0, T]$, $T < \infty$, is more suitable - a decision of the investor or the reader.

A decomposition based on the complete market

We decompose the problem (P)

$$(P) \begin{cases} \max \Psi(x, \pi, C) = E\left[\int_0^T U_1^s(C_s)ds + U_2\left(X^{x,\pi,C}(T)\right)\right] \\ \text{s.t. } (\pi, C) \in D_\pi^*(x) \end{cases}$$

into the static problem (S)

$$(S) \begin{cases} \max \bar{\Psi}(Q, C) = E\left[\int_0^T U_1^s(C_s)dt + U_2(Q)\right] \\ \text{s.t. } (Q, C) \in \mathbb{S} \end{cases}$$

in which $C_s := C(s)$ and

$$\mathbb{S} := \Bigg\{ (Q, C) | Q, C \geq 0,\ Q\ \mathcal{F}_T\text{-measurable},\ C\ \mathcal{F}_t\text{-adapted},$$
$$E\left[\int_0^T \delta(s)C_s ds + \delta(T)Q\right] - x \leq 0\ (\circ),$$
$$E\left[\int_0^T \left(U_1^s(C_s)\right)^- ds + \left(U_2\left(X^{x,\pi,C}(T)\right)\right)^-\right] < \infty\ (\star) \Bigg\}$$

and the representation problem (R)

$$(R) \begin{cases} \text{'Find } (\pi^*, C^*) \in D_\pi^*(x) \text{ with } X^{x,\pi^*,C^*}(T) = Q^*\text{'} \\ \text{s.t. } Q^* \text{ solves (S)} \end{cases}$$

Explanation of the decomposition $(P) \rightarrow (S)$ and (R)

First, we see that C is a consumption process as $(\pi, C) \in \mathbb{S}$.
Moreover, if $(\pi, C) \in D_\pi^*(x)$, theorem 2.7 (Completeness of the market) guarantees that the corresponding capital process $X_t := X^{x,\pi,C}(t)$ satisfies

$$E\left[\int_0^t \delta(s)C_s ds + \delta(t)X_t\right] \leq x \qquad \text{for } t \in I \tag{3.1}$$

especially, X_T holds

$$E\left[\int_0^T \delta(s)C_s ds + \delta(T)X_T\right] \leq x \tag{3.2}$$

Thus, $Q := X_T$ and $C_s = C(s)$ yield $(\circ)$.

In addition to that $(\pi, C) \in D_\pi^*(x)$ implies that $X_T \geq 0$, $C \geq 0$ and $(\star)$ are valid.

Now, theorem 2.7 ii) tells us that there exists a pair $(\pi^*, C^*) \in D_\pi(x)$ with

$$X^{x,\pi^*,C^*}(T) = Q^* \ P\text{-a.s} \tag{3.3}$$

As Q^* fulfills $(\star)$ and is $\mathcal{F}_T$-measurable, we even have $(\pi^*, C^*) \in D_\pi^*(x)$.

This implies that $D_\pi^*(x) = \mathbb{S}$ under the assumption of the completeness of the market model. Hence, (P) and (S) have the same optimizer and this optimizer generates a solution for the representation problem (R).

Please note that the completeness of the market model is evident for this decomposition and consequently for the martingale approach (see (3.1) to (3.3)).

Later, we will see that the optimizer (Q^*, C^*) of (S) yields

$$E\left[\int_0^T \delta(s)C_s^* ds + \delta(T)Q^*\right] = x$$

Furthermore, a solution Q^* can always be found with e.g.:

$$Q^* := \frac{x}{E[\delta(T)]} \text{ and } C_s^* \equiv 0 \qquad \forall\ s \in I\ P\text{-a.s}$$

A heuristic optimization

The Lagrangian multiplier method (see appendix A.4) will help us to solve the problem (P) respectively problems (S) and (R) in a heuristic way. It is heuristic because we imitate this method and besides we treat random variables as (fixed) variables. In the next section 3.3 we will prove that our heuristic solution indeed delivers a maximizer for (P).

First, we consider the problem of maximizing the expected utilitiy of terminal wealth. We check the problem $(\widetilde{S_2})$

$$(\widetilde{S_2}) \begin{cases} \max \widetilde{\Psi_2}(Q) = E[U_2(Q)] \\ \text{s.t. } \underbrace{E[\delta(T)Q] - x}_{=g(Q)\leq 0} \leq 0 \end{cases}$$

for the KKT-conditions.

We temporarily ignore the restrictions $Q \geq 0$, Q $\mathcal{F}_T$-measurable, $(\star)$ and treat Q as a simple variable. Moreover, we set $C(t) \equiv 0 \ \forall\ t \in I$.

We remark that

- $\widetilde{\Psi_2}$ is strictly concave, because U_2 is.
- $\widetilde{\Psi_2} \in \mathcal{C}^1$ as $U_2 \in \mathcal{C}^1$.
- $g \in \mathcal{C}^1$ and g is convex
- A solution for $(\widetilde{S_2})$ with positive Lagrangian multiplier λ is also maximizer for our initial problem (S) (without restriction $(\star)$ and the non-negativity of Q and with $C(t) \equiv 0$) and for the problem $(\hat{S_2})$

$$(\hat{S_2}) \begin{cases} \max \widetilde{\Psi_2}(Q) \\ \text{s.t. } E[\delta(T)Q] - x = 0 \end{cases}$$

The KKT-conditions for $(\widetilde{S_2})$ are

1. $$g(Q) \leq 0 \tag{3.4}$$
2. $$\widetilde{\Psi_2}'(Q) - \lambda \cdot g'(Q) = 0 \tag{3.5}$$
3. $$\lambda \geq 0,\ \lambda \cdot g(Q) = 0 \tag{3.6}$$

We receive from (3.5):

$$\begin{aligned} & \widetilde{\Psi_2}'(Q) - \lambda \cdot g'(Q) = 0 \\ \Rightarrow\ & E[U_2'(Q) - \lambda\delta(T)] = 0 \\ \Rightarrow\ & U_2'(Q) = \lambda\delta(T) > 0 && \text{(see theorem A.10)} \\ \Rightarrow\ & \lambda > 0 && \text{(as } \delta(T) > 0) \\ \Rightarrow\ & \exists!\ I_2 := (U_2')^{-1} && \text{(see theorem A.10)} \\ \Rightarrow\ & Q = I_2(\lambda\delta(T)) > 0 && \text{(see theorem A.10)} \end{aligned}$$

As we have $\lambda > 0$, we know that $g(Q) = 0$, because (3.6) must be valid.

$$\begin{aligned} & g(Q) = 0 \\ \Leftrightarrow\ & \underbrace{E\left[\delta(T) I_2(\lambda\delta(s))\right]}_{=:A_2(\lambda)} = x \\ \Leftrightarrow\ & A_2(\lambda) = x \end{aligned}$$

If we assume that $\exists!\ A_2^{-1}$, then we get as the optimal solution (Q^*):

$$\begin{aligned} & \lambda = A_2^{-1}(A_2(\lambda)) = A_2^{-1}(x) > 0 \\ \Rightarrow \quad & Q^* = I_2\left(A_2^{-1}(x)\delta(T)\right) > 0 \end{aligned}$$

We have found a heuristic solution for the terminal wealth problem $(\widetilde{S_2})$ (maximization of utility of the terminal wealth as a single problem).

Now, we regard the problem $(\widetilde{S_1})$ of maximizing only the utility of consumption which means that our terminal wealth equals zero.

$$(\widetilde{S_1}) \begin{cases} \max \widetilde{\Psi_1}(C) = E\left[\int_0^T U_1^s(C_s)ds\right] \\ \text{s.t. } E\left[\int_0^T \delta(s)C_s ds\right] - x \le 0 \end{cases}$$

Suppose now that the solution of $(\widetilde{S_1})$ C^* is similar to the one of problem $(\widetilde{S_2})$, i.e.:

1. $C_s = I_1^s(\lambda\delta(s))$ with $\lambda > 0$

2. $\underbrace{E\left[\int_0^T \delta(s)I_1^s(\lambda\delta(s))ds\right]}_{=:A_1(\lambda)} = x$

3. $\exists!\ A_1^{-1}$ with $\lambda = A_1^{-1}(x) \quad \Rightarrow \quad C_t^* = I_1^t\left(A_1^{-1}(x)\delta(t)\right) > 0.$

<u>Remark</u>

i) As $\lambda > 0$ C^* solves the problem $(\hat{S}_1)$

$$(\hat{S}_1) \begin{cases} \max \widetilde{\Psi_1}(C) = E\left[\int_0^T U_1^s(C_s)ds\right] \\ \text{s.t. } E\left[\int_0^T \delta(s)C_s ds\right] - x = 0 \end{cases}$$

ii) To see that C^* is optimal we compute

$$\begin{aligned}
\widetilde{\Psi_1}(C^*) &= E\left[\int_0^T U_1^s\left(C_s^*\right)ds\right] \\
&= E\left[\int_0^T U_1^s\left(I_1^s\left(A_1^{-1}(x)\delta(s)\right)\right)ds\right] \quad (\curvearrowright \text{theorem A.10}) \\
&\geq E\left[\int_0^T \left(U_1^s(C_s) + A_1^{-1}(x)\delta(s)\left(C_s^* - C_s\right)\right)ds\right] \\
&= \widetilde{\Psi_1}(C) + \underbrace{A_1^{-1}(x)}_{\geq 0}\left(\underbrace{E\left[\int_0^T C_s^*\delta(s)ds\right]}_{=x} - \underbrace{E\left[\int_0^T C_s\delta(s)ds\right]}_{\leq x}\right) \\
&\geq \widetilde{\Psi_1}(C)
\end{aligned}$$

Thus, we solved the problems $(\widetilde{S_2})$ (respectively $(\widetilde{S_1})$) and $(\hat{S}_2)$ (respectively $(\hat{S}_1)$) heuristically. However, strategies leading to a terminal value without consumption or viceversa a consumption without terminal wealth are not realistic and one-sided. The 'truth' lies in between the problems.

Consequently, we denote:

- $(\bar{S})\begin{cases} \max\ \bar{\Psi}(Q,C) = E\left[\int_0^T U_1^s(C_s)ds + U_2(Q)\right] \\ \text{s.t. } E\left[\int_0^T \delta(s)C_s ds + \delta(T)Q\right] = x \end{cases}$

- $$A(\lambda) = E\left[\int_0^T \delta(s)I_1^s(\lambda\delta(s))ds + \delta(T)I_2(\lambda\delta(T))\right] = x$$

- $$A^{-1}(x) = \lambda \quad \Rightarrow \quad C_t^* = I_1^t\left(A^{-1}(x)\delta(t)\right)$$
$$\Rightarrow \quad Q^* = I_2\left(A^{-1}(x)\delta(T)\right)$$

Now, we have to proof on the hand that (Q^*, C^*) is optimal for (P) and on the other hand that it exists a unique inverse function A^{-1} as described above.

Before we do that we extend our class of utility functions.

3.4 Definition (Extended utility functions)

a) Let $u : (0, \infty) \to \mathbb{R}$ be strictly concave, $u \in \mathcal{C}^1$ s.t.

   i) $u'(0) := \lim_{x \downarrow 0} u'(x) > 0$ and

   ii) $u'(\bar{x}) = 0$ for a unique $\bar{x} \in (0, \infty]$ are held.

   Then u is a ***utility function***.

b) Further $U : [0, T] \times (0, \infty) \to \mathbb{R}$ is a ***utility function***, if it satisfies

   i) $U_t := U(t, \cdot)$ is utility function for any $t \in I$

   ii) $U(\cdot, x) \in \mathcal{C}^0$ for any fixed $x \in (0, \infty)$.

   iii) $U_t'(\bar{x}) := \frac{\partial U}{\partial x}(t, \bar{x}) = 0$ for a unique $\bar{x} \in (0, \infty]$

   iv) $\lim_{x \downarrow 0} U_t'(x) > 0$

Remarks

- Utility functions in terms of definition 3.4 are utility functions in the sense of definition 3.1 with $\bar{x} = \infty$ and $u'(0) = \infty$ or rather $\lim_{x \downarrow 0} U_t'(x) = \infty$.
- Definition 3.4 allows us to deal with a bigger class of utility functions, especially quadratic utility functions (see example 3.5), as often used in finance as criticized for example for the mean-variance-approach.
- From now on we regard utility functions in the sense of definition 3.4.

3.5 Example

With $\alpha, \beta, \gamma \in \mathbb{R}^+$ examples for extended utility functions are:

i) $u(x) = \alpha - \beta \exp(-\gamma x)$

ii) $U(t, x) = \exp(-\alpha t) \cdot -\beta(x - \gamma)^2$

As we change the definition of utility functions, their properties 'change' as well.

3.6 Lemma (Properties of utility functions)

Let u, U be utility functions. Then we have

1. u, U are strictly increasing on $(0, \bar{x}]$
2. u', U_t' are strictly decreasing on $[\bar{x}, \infty)$
3. $u' \in \mathcal{C}^0$ is strictly decreasing for $x \in [0, \bar{x}]$,
   $u' : [0, \bar{x}] \to [0, u'(0)]$
4. U_t' is strictly decreasing for $x \in [0, \bar{x}]$ and any fixed $t \in I$,
   $U_t' : \{t\} \times [0, \bar{x}] \to [0, U_t'(0)]$

Proof: is similar to the one in appendix A.3.

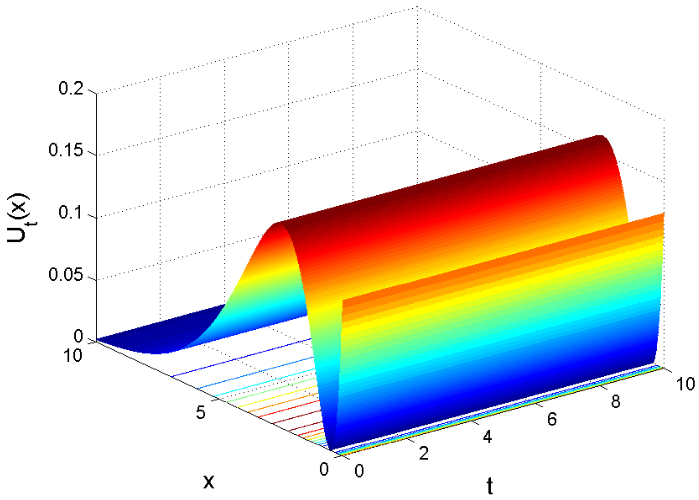

Figure 3.2: Extended utility function $U_t(x) = \exp(-t) \cdot -\frac{1}{2}(x - \frac{1}{2})^2$

3.7 Corollary (Properties of I_1^t and I_2)
Let be I_1^t and I_2 the unique inverse functions of $(U_1^t)'$ and U_2'. Hence, we get

1. I_1^t, I_2 are strictly decreasing on $[0, (U_1^t)'(0)]$ or rather $[0, U_2'(0)]$
2. $I_1^t \in \mathcal{C}^0[0, (U_1^t)'(0)]$
3. $I_1^t : [0, (U_1^t)'(0)] \to [0, x_1]$, i.e. $\lim_{y \to 0} I_1^t(y) = x_1$ and $\lim_{y \to (U_1^t)'(0)} I_1^t(y) = 0$
4. $I_2 \in \mathcal{C}[0, U_2'(0)]$
5. $I_2 : [0, U_2'(0)] \to [0, x_2]$, i.e. $\lim_{y \to 0} I_2(y) = x_2$ and $\lim_{y \to U_2'(0)} I_2(y) = 0$

Proof: follows from theorem A.10 and the preceding lemma 3.6.

Notation

We extend the inverse functions I_1^t and I_2 as follows:

$$I_1^t(y) := \begin{cases} I_1^t(y) & \text{if } y \in [0, (U_1^t)'(0)] \\ 0 & \text{else} \end{cases}$$

$$I_2(y) := \begin{cases} I_2(y) & \text{if } y \in [0, U_2'(0)] \\ 0 & \text{else} \end{cases}$$

Remarks

- As extended versions we see that I_1^t and $I_2 \in \mathcal{C}^0[0, \infty)$.
- I_1^t and I_2 are strictly decreasing on $(0, \infty)$ under the additional assumption $\int_0^T \|\xi(s)\|^2 ds < \infty$ (compare Korn [2], pp. 65-67).

With these observations we analyze the term

$$A(\lambda) = E\left[\int_0^T \delta(s) I_1^s(\lambda\delta(s))ds + \delta(T) I_2(\lambda\delta(T))\right] \tag{3.7}$$

3.8 Lemma (Properties of $A(\lambda)$)
Assume

- (3.7),
- $A(\lambda) < \infty \qquad \forall\, \lambda \in (0, \infty)$,
- $(U_1^t)'(0) < \infty\ \forall\, t \in I$ and $U_2'(0) < \infty$,
- $\int_0^T \|\xi(s)\|^2 ds < \infty$.

This implies

i) $A(\lambda) \in \mathcal{C}^0(0, \infty)$,

ii) $A(\lambda)$ is strictly decreasing on $(0, \infty)$,

iii) $A(\infty) := \lim_{\lambda \to \infty} A(\lambda) = 0$

iv)

$$A(0) := \lim_{\lambda \to 0} A(\lambda) = \begin{cases} \infty & \text{if } \lim_{x \to \infty} (U_1^t)'(x) = 0\ \forall\, t \in I \vee \lim_{x \to \infty} U_2'(x) = 0 \\ x_1 E\left[\int_0^T \delta(s) ds\right] + x_2 E\left[\delta(T)\right] & \end{cases}$$

in which $(U_1^t)'(x_1) = 0\ \forall\, t \in I$ and $U_2(x_2) = 0$.

Sketch of proof

i) $A \in \mathcal{C}^0(0, \infty)$:

- $\delta \in \mathcal{C}^0(0, \infty)$
- I_1^t, $I_2 \in \mathcal{C}^0(0, \infty)$ (as extended versions)
- use the theorem of dominated convergence A.4, appendix A.2

$\Rightarrow\ A \in \mathcal{C}^0(0, \infty)$

ii) As I_1^t and I_2 are strictly decreasing, A is strictly decreasing, too.

iii) $\lim_{y \to \infty} I_1^t(y) = \lim_{y \to \infty} I_2(y) = 0\ \forall\, t \in I$.
Moreover, I_1^t and I_2 are strictly decreasing and $A(\lambda) < \infty\ \forall\, \lambda \in (0, \infty)$ is satisfied.
With the theorem of monotone convergence A.5, appendix A.2 it follows:

$$\lim_{\lambda \to \infty} A(\lambda) = 0$$

iv)

- $\lim\limits_{x\to\infty} (U_1^t)'(x) = 0 \ \forall \ t \in I, \ I_1^t, \ I_2 > 0$
  apply the the lemma of Fatou A.6, appendix A.2:

$$\Rightarrow \quad \liminf_{\lambda\to 0} A(\lambda) \geq \liminf_{\lambda\to 0} E\left[\int_0^T \delta(s) I_1^s(\lambda\delta(s))ds\right]$$

$$\geq E\left[\int_0^T \delta(s) \underbrace{\liminf_{\lambda\to 0} I_1^s(\lambda\delta(s))}_{=\infty} ds\right]$$

$$= \infty$$

- $\lim\limits_{x\to\infty} U_2'(x) = 0, \ I_1^t, \ I_2 > 0$
  apply the the lemma of Fatou A.6, appendix A.2:

$$\Rightarrow \quad \liminf_{\lambda\to 0} A(\lambda) \geq \liminf_{\lambda\to 0} E\left[\delta(T) I_2(\lambda\delta(T))\right]$$

$$\geq E[\delta(T) \underbrace{\liminf_{\lambda\to 0} I_2(\lambda\delta(T))}_{=\infty}]$$

$$= \infty$$

- For $\lim\limits_{x\to\infty} U_2'(x) \neq 0$ and $\lim\limits_{x\to\infty} (U_1^t)'(x) \neq 0 \ \forall \ t \in I$ we get:

$$\limsup_{\lambda\to 0} A(\lambda) = \limsup_{\lambda\to 0} E\left[\int_0^T \delta(s) \underbrace{I_1^s(\lambda\delta(s))}_{\leq x_1} ds + \delta(T) \underbrace{I_2(\lambda\delta(T))}_{\leq x_2}\right]$$

$$\leq x_1 E\left[\int_0^T \delta(s)ds\right] + x_2 E[\delta(T)]$$

$$\liminf_{\lambda\to 0} A(\lambda) = \liminf_{\lambda\to 0} E\left[\int_0^T \delta(s) I_1^s(\lambda\delta(s))ds + \delta(T) I_2(\lambda\delta(T))\right]$$

$$(\curvearrowright \text{Fatou}) \ \geq E\left[\int_0^T \delta(s) \underbrace{\liminf_{\lambda\to 0} I_1^s(\lambda\delta(s))}_{=x_1} ds\right]$$

$$+ E\left[\delta(T) \underbrace{\liminf_{\lambda\to 0} I_2(\lambda\delta(T))}_{=x_2}\right]$$

$$= x_1 E\left[\int_0^T \delta(s)ds\right] + x_2 E[\delta(T)]$$

■

Now, we define

$$\hat{x} := \begin{cases} x_1 E\left[\int_0^T \delta(s)ds\right] + x_2 E[\delta(T)] & \text{if } x_1 < \infty \text{ and } x_2 < \infty \\ \infty & \text{else} \end{cases} \quad (3.8)$$

Thus, we get the important result:

3.9 Corollary (Existence and uniqueness of $A^{-1}(\lambda)$)
With $A(\lambda) : [0,\infty] \to [0,\hat{x}]$ as in lemma 3.8 and $\hat{x}$ as in (3.8) we have

$\exists!\ \Gamma := A^{-1} : [0,\hat{x}] \to [0,\infty]$ s.t.

i) $\Gamma \in \mathcal{C}^0$ and

ii) Γ is strictly decreasing

Proof
Corollary 3.9 is a direct implication of lemma 3.8. ∎

3.3 Optimal portfolios

Besides theorem 2.7, the completeness of the market, the next theorem is the principal one in this thesis.

We suppose that the investor is endowed with an initial capital x, $x \in \mathbb{R}^+$, and that the assumptions of lemma 3.8, properties of $A(\lambda)$, are satisfied,
i.e. (3.7) is valid and

$$A(\lambda) < \infty \quad \forall\, \lambda \in (0,\infty) \tag{3.9}$$

$$(U_1^t)'(0) < \infty \quad \forall\, t \in I \text{ and } U_2'(0) < \infty \tag{3.10}$$

$$\int_0^T \|\xi(s)\|^2 ds < \infty \tag{3.11}$$

Further, let $\hat{x}$ be defined as in (3.8) with $(U_1^t)'(x_1) = 0$ as property for $x_1 \in (0,\infty]$ and $U_2'(x_2) = 0$ for $x_2 \in (0,\infty]$.

3.10 Theorem (Optimal portfolios)
With the preceding assumptions (3.7), (3.9) - (3.11) we get the following results for the portfolio problem (P) :

1. The optimal terminal wealth Q^* is given by

$$Q^* := \begin{cases} x_2, & \text{if } x \geq \hat{x} \\ I_2(A^{-1}(x)\delta(T)), & \text{else} \end{cases} \tag{3.12}$$

2. The optimal consumption process C_t^*, $t \in I$, is given by

$$C_t^* := \begin{cases} x_1, & \text{if } x \geq \hat{x} \\ I_1^t(A^{-1}(x)\delta(t)), & \text{else} \end{cases} \tag{3.13}$$

3. It exists $x^* \in [0,x]$ and a corresponding portfolio process $\pi^*(t)$, $t \in I$, s.t.

   i)
   $$(\pi^*(t), C_t^*) \in D_\pi^*(x^*)$$

   ii)
   $$X^{x^*,\pi(t)^*,C_t^*}(T) = Q^* \quad P\text{-a.s.}$$

   iii)
   $$\Psi\left(x^*, \pi^*(t), C_t^*\right) = \max_{(\pi(t),C_t) \in D_\pi^*(z)} \Psi(z, \pi(t), C_t) \quad \text{for } z \leq x$$

   iv) If (3.10) is not valid, we have $x^* = x$.

Proof: see appendix B.
The existence of a solution results from theorem 2.7. With help of a case differentiation ($x \geq \hat{x}$ and $x < \hat{x}$) show that $\max U_1^t(\cdot) = x_1$ and $\max U_2(\cdot) = x_2$ are path-wise optimal for the first case and use the properties of C_t^*, Q, I_1^t, I_2, A^{-1} and δ for the second one. The optimality is derived as in the heuristic solution.

Interpretation of theorem 3.10

- Theorem 3.10 tells us that the initial value x is not necessarily completely invested into the portfolio if the investor deals with utility functions in terms of definition 3.4 $\rightsquigarrow x^* \in [0, x]$. x^* is consequently the invested capital.
- If utility functions according to definition 3.1 are used, the initial value is equal to the invested capital (i.e. the *complete* capital x is put into the portfolio).
- For utility functions with unique nulls x_1 and x_2 of their derivatives (with x_1, $x_2 \in (0, \infty)$) the optimal consumption and optimal terminal value depend on the invested capital x.
  If x exceeds $\hat{x}$ then we set $Q^* = x_2$ and $C_t^* = x_1$ (if this is financiable for the investor which means $x \geq \hat{x}$).
  If one of the utility functions yields $U_i(x_i)$ with $x_i = \infty$ $(i = 1, 2)$ or if we have $x < \hat{x}$, the solution with the inverse function I_1, I_2 has to be used.
  This fact can be explained as follows:
  Suppose one of the nulls x_i $(i = 1, 2)$ is finite. The theorem shows that it is not optimal to choose x_i as value for the consumption process $(i = 1)$ or for the terminal value $(i = 2)$. As the consumption rate / terminal value is close to x_i the additional utility of additional capital for consumption / terminal wealth converges to zero. Hence, this additional amount can be invested into terminal capital / consumption and implies that the additional utility is higher than in the other (above mentioned) case.

The next corollaries are directly deduced from theorem 3.10.

3.11 Corollary (Optimal consumption problem)
The problem (P_1)

$$(P_1)\begin{cases} \max\ E\left[\int_0^T U_1^s(C_s)ds\right] \\ \text{s.t. } (\pi, C) \in D_\pi^*(x) \end{cases}$$

is solved by

$$C_t^* := \begin{cases} x_1, & \text{if } x \geq \hat{x} \\ I_1^t(A^{-1}(x)\delta(t)), & \text{else} \end{cases}$$

and it exists $x^* \in [0, x]$ and a corresponding portfolio process $\pi^*(t)$, $t \in I$, s.t.

i)
$$(\pi^*(t), C_t^*) \in D_\pi^*(x^*)$$

ii)
$$X^{x^*, \pi(t)^*, C_t^*}(T) = 0 \quad P\text{-a.s.}$$

Proof
Set $U_2 \equiv 0$ in theorem 3.10. ■

3.12 Corollary (Optimal terminal wealth problem)
The problem (P_2)

$$(P_2)\begin{cases} \max E\left[U_2(X^{x,\pi,C}(T))\right] \\ \text{s.t. } (\pi,0)\in D^*_\pi(x) \end{cases}$$

is solved by

$$Q^* := \begin{cases} x_2, & \text{if } x \geq \hat{x} \\ I_2(A^{-1}(x)\delta(T)), & \text{else} \end{cases}$$

It exists $x^* \in [0,x]$ and a corresponding portfolio process $\pi^*(t)$, $t\in I$, s.t.

i)
$$(\pi^*(t),0)\in D^*_\pi(x^*)$$

ii)
$$X^{\pi(t)^*}(T) = Q^* \quad P\text{-a.s.}$$

Proof
Set $U_1^s \equiv 0$ in theorem 3.10. ∎

We have found an optimizer for the static problem (S), but we also need to find an answer for the representation problem (R).
We will consider an example whose solution will be used afterwards to explain the approach to resolve (R).

3.13 Example
Let be $U_1(x) := U_1^t(x) = \ln(x)$ and $U_2(x) = \ln(x)$.

$$\begin{aligned}
\Rightarrow \quad & U_1'(x) = U_2'(x) = \frac{1}{x} \\
& U_i(x) = y \Leftrightarrow \frac{1}{x} = y \quad \Rightarrow x = \frac{1}{y} = I_i(y) \quad \Rightarrow \hat{x} = \infty \\
\Rightarrow \quad & A(\lambda) = E\left[\int_0^T \delta(s) I_1(\lambda\delta(s))ds + \delta(T) I_2(\lambda\delta(T))\right] \\
& \quad = E\left[\int_0^T \delta(s)\frac{1}{\lambda\delta(s)}ds + \delta(T)\frac{1}{\lambda\delta(T)}\right] = \frac{1}{\lambda}\left(\int_0^T 1ds + 1\right) \\
& \quad = \frac{1}{\lambda}(T+1) \\
& A(\lambda) = x \Leftrightarrow \frac{1}{\lambda}(T+1) = x \quad \Rightarrow \lambda = \frac{1}{x}(T+1) =: A^{-1}(x)
\end{aligned}$$

With theorem 3.10 we obtain:

$$\begin{aligned} Q^* &= I_2\left(A^{-1}(x)\delta(T)\right) \\ &= I_2\left(\frac{1}{x}(T+1)\delta(T)\right) \\ &= \frac{x}{(T+1)\delta(T)} \end{aligned} \tag{3.14}$$

$$\begin{aligned} C^* &= I_1^t\left(A^{-1}(x)\delta(t)\right) \\ &= I_1^t\left(\frac{1}{x}(T+1)\delta(t)\right) \\ &= \frac{x}{(T+1)\delta(t)} \end{aligned} \tag{3.15}$$

Excursion

We show that the function g is indeed convex as stipulated.

$$\begin{aligned} Q &= \frac{x}{(T+1)\delta(T)} \quad \Leftrightarrow \quad x = Q\cdot\delta(T)(T+1) \\ \Rightarrow \quad g(Q) &= E\left[\frac{x}{(T+1)\delta(T)}\delta(T)\right] - x \\ &= \frac{x}{T+1} - x = \frac{(T+1)\delta(T)\cdot Q}{T+1} - Q\cdot\delta(T)(T+1) \\ &= -T\delta(T)\cdot Q \end{aligned}$$

According to definition A.9, appendix A.3, $g(Q)$ is convex.

To get a representation for the optimal portfolio process $\pi^*(t)$ we define $X(t) := f(t, g(t))$ with

i) $g(t) := \int_0^t \underbrace{\left(r(s) + \frac{1}{2}\|\xi(s)\|^2\right)}_{=K_s} ds + \int_0^t \underbrace{\xi(s)^t}_{=L_s} dB_s$

ii) $f(t,z) := x\left(1 - \frac{t}{T+1}\right)\exp(z)$

We see that

i) $\exp(g(t)) = \frac{1}{\delta(t)}$

ii) $X(0) = x$

iii) $X(T) = X(T, g(T)) = x\left(\frac{T+1-T}{T+1}\right)\exp(g(T)) = \frac{x}{(T+1)\delta(T)} = Q^*$

Now, we apply the multi-dimensional Itô formula (theorem A.8, appendix A, $n = m = 1$) for $X(t) = f(t,z)$:

$$dX(t) = \left(\frac{\partial f}{\partial t}(t,g(t)) + \frac{\partial f}{\partial z}(t,g(t))K_t + \frac{1}{2}\frac{\partial^2 f}{\partial^2 z}(t,g(t))L_t^2\right)dt + \frac{\partial f}{\partial z}(t,g(t))L_t dB_t$$

Auxiliary calculations

i)

$$\begin{aligned}\frac{\partial f}{\partial t}(t,g(t)) &= -\frac{x}{T+1}\exp(g(t)) = x\frac{T+1-t}{T+1}\exp(g(t))\left(-\frac{1}{T+1-t}\right)\\ &= X(t)\left(-\frac{1}{T+1-t}\right)\end{aligned}$$

ii)

$$\frac{\partial f}{\partial z}(t,g(t)) = \frac{\partial^2 f}{\partial^2 z}(t,g(t)) = x\left(1-\frac{t}{T+1}\right)\exp(g(t)) = X(t)$$

$$\begin{aligned}\Rightarrow \quad dX(t) &= X(t)\left[\left(-\frac{1}{T+1-t}+r(t)+\frac{1}{2}\|\xi(t)\|^2+\frac{1}{2}\|\xi(t)\|^2\right)dt+\xi(t)^t dB_t\right]\\ &= X(t)\left[\left(-\frac{1}{T+1-t}+r(t)+\|\xi(t)\|^2\right)dt+\xi(t)^t dB_t\right] \qquad (3.16)\end{aligned}$$

with $X(0)=x$. We remember the general representation (CE) for the wealth process from chapter 2.2:

$$dX(t) = X(t)\left(r(t)+\pi(t)^t(b(t)-r(t)e^n)\right)dt - C(t)dt + X(t)(\pi(t)^t\sigma(t))dB_t$$

By comparing the coefficients we see that

i)

$$\begin{aligned} & \xi(t)^t = \pi(t)^t\sigma(t)\\ \Leftrightarrow \quad & \pi(t)^t = \xi(t)^t\sigma(t)^{-1}\\ \Leftrightarrow \quad & \pi^*(t) = \left(\sigma(t)^{-1}\right)^t\xi(t) \qquad (1)\end{aligned}$$

ii) with respect to (1) in (CE):

$$\begin{aligned} & X(t)\left(r(t)+\xi(t)^t\underbrace{\sigma(t)^{-1}(b(t)-r(t)e^n)}_{=\xi(t)}\right) - C(t)\\ & = X(t)\left(-\frac{1}{T+1-t}+r(t)+\|\xi(t)\|^2\right)\\ \Leftrightarrow \quad & X(t)\|\xi(t)\|^2 - C(t) = X(t)\left(-\frac{1}{T+1-t}+\|\xi(t)\|^2\right)\\ \Rightarrow \quad & C(t)^* = \left(\frac{1}{T+1-t}\right)X(t) \qquad (2)\end{aligned}$$

Hence, with $(\pi(t)^*, C(t)^*)$ as in (1) and (2) the (CE) for the wealth process is satisfied. Moreover, $X(t)$ is positive as $f(t,g(t))$ is and $(\pi^*, C^*) \in D_\pi^*(x)$. Thus, we have found an optimal consumption and portfolio process and the corresponding capital process.

We remark that 3.14 and (2) must be equal because they are both representations for the (optimal) consumption process $C(t)^*$. This implies

$$\begin{aligned} \frac{x}{(T+1)\delta(t)} &= \frac{1}{T+1-t}X(t) \\ \Leftrightarrow \quad X(t) &= x\frac{T+1-t}{T+1}\frac{1}{\delta(t)} = x\left(1-\frac{t}{T+1}\right)\exp(g(t)) \qquad \surd \end{aligned}$$

(2) has the advantage to show the direct dependence of $C(t)^*$ from $X(t)$.

In the case of one stock and constant market coefficients r, b and σ we obtain $\pi \equiv \frac{b-r}{\sigma^2}$. This term is often called ***market price of risk*** and is closely connected with the Sharpe ratio $\frac{b-r}{\sigma}$ (see explanation of the δ-factor, chapter 2.3).

Please note that even if $\pi \equiv \frac{b-r}{\sigma^2}$ is constant, the portfolio has to be redistributed at every time instant $t \in I$.

3.4 Comparison of coefficients

To solve the representation problem (R) we summarize the procedure in example 3.13:

First, we computed the optimal terminal wealth Q^* and consumption process C_t^* with help of the inverse functions I_1^t, I_2 and $\Gamma := A^{-1}$. Then, we 'guessed' a process $X(t)$ with $X(0) = x$ and $X(T) = Q^*$ and wrote $X(t)$ as a functional in dependence of the Brownian motion and the market coefficients ($f(t, g(t))$ in example 3.13).

Furthermore, we applied the Itô formula to this functional and received a SDE. The coefficients of this SDE were compared to the ones of the capital equation (CE), the general form of a wealth process.

For the resulting portfolio process π^* it was checked if

i) $(\pi^*(t), C_t^*) \in D_\pi^*(x)$ (i.e $X^{x,\pi^*,C^*}(T)$ is admissible) and

ii) π^* indeed generates π^* and C_t^*.

Thus, we obtained an optimal pair (π^*, C^*) which corresponded with the capital process $X(t)$.

However, 'guessing' a process is not really mathematical. We take a closer look at the proof of theorem 2.7 (see appendix B) and notice that in fact the 'guess' is the computation of the conditional expectation of

$$\frac{1}{\delta(t)}E\left[\int_t^T \delta(s)C_s^*ds + \delta(T)Q^*|\mathcal{F}_t\right]$$

All in all, this approach is called ***comparison of coefficients*** and is formalized as follows:

3.14 Theorem (Solution of (R))
Assumptions:

- the complete market settings
- $$X(t) = \frac{1}{\delta(t)} E\left[\int_t^T \delta(s) C_s^* ds + \delta(T) Q^* | \mathcal{F}_t\right] = f(t, B_1(t), \ldots, B_n(t)) \quad (3)$$

  in which $f \geq 0$, $f \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n)$ with $f(0, \ldots, 0) = x$.
- Q^* the optimal terminal wealth resulting from theorem 3.10 for the problem (P).
- C_t^* the optimal consumption process resulting from theorem 3.10 for the problem (P).
- x^* the initial value resulting from theorem 3.10 for the problem (P).

Then, the optimal trading strategy $\vartheta^*(t) = (\vartheta_0^*(t), \ldots, \vartheta_n^*(t))^t$, $t \in I$ is given by

$$\vartheta_i^*(t) = \frac{1}{P_i(t)} \left(\left(\sigma(t)^{-1}\right)^t \cdot \nabla_B f(t, B_1(t), \ldots, B_n(t))^t \right)_i \qquad i = 1, \ldots, n \qquad (3.17)$$

$$\vartheta_0^*(t) = \frac{X(t) - \sum_{i=1}^{n} \vartheta_i^*(t) P_i(t)}{P_0(t)} \qquad (3.18)$$

$\rightarrow$ $X(t)$ is the corresponding wealth process to (ϑ^*, C_t^*) as long as ϑ^* satisfies the assumptions of definition 2.2.

$\rightarrow$

$$\nabla_B f(t, B_1(t), \ldots, B_n(t)) := \begin{pmatrix} \frac{\partial f}{\partial B_1}(t, B_1(t), \ldots, B_n(t)) \\ \vdots \\ \frac{\partial f}{\partial B_n}(t, B_1(t), \ldots, B_n(t)) \end{pmatrix}$$

Further, the optimal portfolio process $\pi^*(t) = (\pi_1^*(t), \ldots, \pi_n^*(t))^t$ of theorem 3.10 is given by

$$\pi^*(t) = \frac{1}{X(t)} \left(\sigma(t)^{-1}\right)^t \cdot \nabla_B f(t, B_1(t), \ldots, B_n(t))^t \qquad (3.19)$$

Proof: see appendix B.
Apply the Itô formula A.8 for

$$f(t, B_1(t), \ldots, B_n(t)) = \frac{1}{\delta(t)} E\left[\int_t^T \delta(s) C_s^* ds + \delta(T) Q^* | \mathcal{F}_t\right] = X_t$$

and compare the integrands of the stochastic integrals with those of the representation of X_t in (2.11).

Remarks

- There is one great restriction in this theorem: a representation of the conditional expectation as it is specified in (3) is only possible if the market coefficients are constant on the interval $[0, t]$. For general market coefficients and an analogous theorem deep functional analytic methods would be required, see Occone and Karatzas [14].
- Q and C_t do not need to be optimal in the sense of theorem 3.10. $(Q, C_t) \geq 0$ with $E\left[\int_0^T \delta(s)C_s ds + \delta(T)Q\right] = x^* > 0$ allows to apply theorem 2.7, part b) and generates a portfolio process π or respectively a strategy ϑ with help of theorem 3.10. If they are optimal is not sure, of course.

In example 3.13 we had
$f(t, g(t)) = x\left(1 - \frac{t}{T+1}\right)\exp(g(t))$ with
$g(t) = \int_0^t \left(r(s) + \frac{1}{2}\|\xi(s)\|^2\right) ds + \int_0^t \xi(s)^t dB_s.$

We apply theorem 3.14 and get

$$\nabla_B f(t, B) = x\left(1 - \frac{t}{T+1}\right)\exp(g(t)) \cdot \frac{\partial g}{\partial B}(B) = X(t)\xi(t)^t$$
$$\Rightarrow \quad \underline{\pi^*(t) = \left(\sigma(t)^{-1}\right)^t \xi(t)}$$

as we have already seen in 3.13.

3.5 A solution for the portfolio problem

As seen we know by now the procedure to solve the problem (P) from the preceding sections. We summarize the different steps in the following algorithm.

3.15 Algorithm
Suppose the initial capital x and the utility functions U_1^t and U_2 are given.

Step 1
Determine x_1, x_2 and $\hat{x}$ with help of U_1^t and U_2 and from equation (3.8).

Step 2
Compute I_1^t and I_2 resulting from the derivatives of U_1^t and U_2.

Step 3
Deduce A^{-1} from I_1^t, I_2 and equation (3.7).

Step 4
Calculate Q^* and C_t^* as in theorem 3.10 (see equations (3.12) and (3.13)).

Step 5

With help of Q^* and C_t^* solve the conditional expectation

$$\frac{1}{\delta(t)} E\left[\int_t^T \delta(s) C_s^* ds + \delta(T) Q^* | \mathcal{F}_t\right] = X(t)$$

Step 6

Find a function as declared in theorem 3.14 such that

$$X(t) = f(t, B_1(t), \ldots, B_n(t))$$

is satisfied (f is computed in step 5).

Step 7

Compute $\nabla_B f(t, B_1(t), \ldots, B_n(t))$ and determine $\vartheta_i^*(t)$, $i = 0, \ldots, n$, and $\pi^*(t)$ as in theorem 3.14 (see equations (3.17) to (3.19)).

Step 8

i) Put C_t^* and $\pi^*(t)$ into the capital process $X(t)$.
ii) Set $x^* := X(0)$ if (3.10) is valid, otherwise $x^* := x = X(0)$.

Result

- the optimal invested capital x^*
- the optimal terminal value Q^*
- the optimal consumption process C_t^*
- the optimal trading strategy $\vartheta^*(t)$
- the optimal portfolio process $\pi^*(t)$
- the resulting capital process $X^{x^*, \pi^*(t), C_t^*}(t)$, $t \in I$

4 Applications and numerical tests

In this chapter we want to apply the deduced procedure (algorithm 3.15) to an example which is similar to the one of section 3.3, see example 3.13.

4.1 Logarithmic utility

Define the utility functions U_1^t and U_2 as

$$U_1^t(x) = \exp(-\alpha t)\ln(x) \text{ and } U_2(x) = ln(x)$$

with $\alpha \in \mathbb{R}^+$.

Now, we want to compute a general solution for the portfolio problem $(\check{P})$ for the initial value x with the above mentioned algorithm.

$$(\check{P}) \begin{cases} \max \ \Psi(x,\pi,C) = E\left[\int_0^T \exp(-\alpha s)\ln(C_s)ds + \ln\left(X^{x,\pi,C}(T)\right)\right] \\ \text{s.t. } (\pi, C) \in D_\pi^*(x) \end{cases}$$

Step 1

$$U_1^t(x) = \exp(-\alpha t)\ln(x) \quad \Rightarrow \quad \left(U_1^t\right)'(x) = \frac{\exp(-\alpha t)}{x} \quad \Rightarrow \quad x_1 = \infty$$

$$U_2(x) = \ln(x) \quad \Rightarrow \quad U_2'(x) = \frac{1}{x} \quad \Rightarrow \quad x_2 = \infty \quad \Rightarrow \quad \hat{x} = \infty$$

Step 2

As x, $y > 0$ we have

$$y = \frac{\exp(-\alpha t)}{x} \quad \Rightarrow \quad \frac{\exp(-\alpha t)}{y} = I_1^t(y) \quad \text{and} \quad y = \frac{1}{x} \quad \Rightarrow \quad \frac{1}{y} = I_2(y)$$

Step 3

$$\begin{aligned} A(\lambda) &= E\left[\int_0^T \delta(s) I_1^s(\lambda\delta(s))ds + \delta(T) I_2(\lambda\delta(T))\right] \\ &= E\left[\int_0^T \delta(s)\frac{\exp(-\alpha s)}{\lambda\delta(s)}ds + \delta(T)\frac{1}{\lambda\delta(T)}\right] \\ &= \frac{1}{\lambda}E\left[\int_0^T \exp(-\alpha s)ds + 1\right] \\ &= \frac{1}{\lambda}\left(\int_0^T \exp(-\alpha s)ds + 1\right) = \frac{1}{\lambda}\left(-\frac{1}{\alpha}\exp(-\alpha T) + \frac{1}{\alpha} + 1\right) \\ &= \frac{1}{\lambda\alpha}\left(-\exp(-\alpha T) + 1 + \alpha\right) \quad (\overset{!}{=} x) \end{aligned}$$

$$\Rightarrow \quad A^{-1}(x) = \frac{1}{x\alpha}\left(-\exp(-\alpha T) + 1 + \alpha\right) \quad (= \lambda) \qquad \text{as } x,\, \lambda > 0.$$

Step 4

$$\begin{aligned} Q^* &= I_2\left(A^{-1}(x)\delta(T)\right) = \frac{x\cdot\alpha}{\alpha+1-\exp(-\alpha T)}\cdot\frac{1}{\delta(T)} \\ C_t^* &= I_1^t\left(A^{-1}(x)\delta(t)\right) = \frac{x\cdot\alpha\cdot\exp(-\alpha t)}{\alpha+1-\exp(-\alpha T)}\cdot\frac{1}{\delta(t)} \end{aligned}$$

Step 5

$$\begin{aligned} X(t) =& \frac{1}{\delta(t)}E\left[\int_t^T \delta(s)C_s^* ds + \delta(T)Q^*|\mathcal{F}_t\right] \\ =& \frac{1}{\delta(t)}E\left[\int_t^T \frac{x\cdot\alpha\cdot\exp(-\alpha s)}{\alpha+1-\exp(-\alpha T)}\cdot\frac{\delta(s)}{\delta(s)}ds + \frac{x\cdot\alpha}{\alpha+1-\exp(-\alpha T)}\cdot\frac{\delta(T)}{\delta(T)}|\mathcal{F}_t\right] \\ =& \frac{1}{\delta(t)}\cdot\frac{x\cdot\alpha}{\alpha+1-\exp(-\alpha T)}\cdot E\left[\int_t^T \exp(-\alpha s)ds + 1|\mathcal{F}_t\right] \\ =& \frac{1}{\delta(t)}\cdot\frac{x\cdot\alpha}{\alpha+1-\exp(-\alpha T)}\cdot E\left[-\frac{1}{\alpha}\exp(-\alpha T) + \frac{1}{\alpha}\exp(-\alpha t) + 1|\mathcal{F}_t\right] \\ =& \frac{1}{\delta(t)}\cdot\frac{x\cdot\alpha}{\alpha+1-\exp(-\alpha T)}\cdot\frac{1}{\alpha}\left(\alpha+\exp(-\alpha t)-\exp(-\alpha T)\right)\cdot\underbrace{E[1|\mathcal{F}_t]}_{=1} \\ =& x\cdot\frac{\alpha+\exp(-\alpha t)-\exp(-\alpha T)}{\alpha+1-\exp(-\alpha T)}\cdot\frac{1}{\delta(t)} \end{aligned}$$

We verify: $X(0) = x$ and $X(T) = Q^*$. $\checkmark$

Moreover, we get C_t in dependence on $X(t)$ with

$$C_t^* = \frac{\alpha\exp(\alpha t)}{\alpha+\exp(-\alpha t)-\exp(-\alpha T)}\cdot X(t)$$

Step 6

$$\begin{aligned} X(t) &= \underbrace{\frac{x}{\alpha+1-\exp(-\alpha T)}}_{=:\eta}\cdot\underbrace{\left(\delta(t)^{-1}(\alpha+\exp(-\alpha t)-\exp(-\alpha T))\right)}_{=:g(t,B_1(t),\ldots,B_n(t))} \\ &= \eta\cdot g(t,B_1(t),\ldots,B_n(t)) \\ &=: f(t,B_1(t),\ldots,B_n(t)) \end{aligned}$$

in which

$$\begin{aligned} & g(t,B_1(t),\ldots,B_n(t)) \\ =& (\alpha+\exp(-\alpha t)-\exp(-\alpha T))\exp\left(\int_0^t\left(r(s)+\frac{1}{2}\|\xi(s)\|^2\right)ds + \xi(s)^t dB_s\right) \end{aligned}$$

Step 7

$$\begin{aligned}
& \nabla_B f(t, B_1(t), \ldots, B_n(t)) \\
&= \eta \cdot \nabla_B g(t, B_1(t), \ldots, B_n) \\
&= \eta \cdot g(t, B_1(t), \ldots, B_n(t)) \cdot \xi^t \\
&= X(t) \cdot \xi(t)^t \\
\Rightarrow \quad \vartheta_i^*(t) &= \frac{X(t)}{P_i(t)} \cdot \left(\left(\sigma(t)^{-1}\right)^t \cdot \xi(t) \right)_i \\
\Rightarrow \quad \vartheta_0^*(t) &= \frac{X(t) - \sum_{i=1}^n \vartheta_i^*(t) \cdot P_i(t)}{P_0(t)} \\
&= \frac{X(t)}{P_0(t)} \cdot \left(1 - \sum_{i=1}^n \left(\left(\sigma(t)^{-1}\right)^t \cdot \xi(t) \right)_i \right) \\
\Rightarrow \quad \pi^*(t) &= \frac{1}{X(t)} \cdot \left(\sigma(t)^{-1}\right)^t \cdot X(t) \cdot \xi(t) = \left(\sigma(t)^{-1}\right)^t \cdot \xi(t)
\end{aligned}$$

Step 8

$$\begin{aligned}
dX(t) &= (X(t)r(t) - C(t))dt + X(t) \left(\pi^*(t)\right)^t \left(b(t) - r(t)e^n\right) dt \\
&\quad + X(t)\pi^*(t)^t \sigma dB_t \\
&= \Big[X(t)r(t) - \frac{x \cdot \alpha \cdot \exp(-\alpha t)}{\alpha + 1 - \exp(-\alpha T)} \cdot \frac{\alpha + \exp(-\alpha t) - \exp(-\alpha T)}{\alpha + \exp(-\alpha t) - \exp(-\alpha T)} \delta(t)^{-1} \\
&\quad + X(t) \underbrace{\xi(t)^t \sigma(t)^{-1} \left(b(t) - r(t)e^n\right)}_{=\|\xi(t)\|^2} \Big] dt + X(t)\xi(t)^t dB_t \\
&= X(t) \left(\left(r(t) - \frac{\alpha \exp(-\alpha t)}{\alpha + \exp(-\alpha t) - \exp(-\alpha T)} + \|\xi(t)\|^2 \right) dt + \xi(t)^t dB_t \right)
\end{aligned}$$

in which

$$\begin{aligned}
X(t) = x \cdot & \frac{\alpha + \exp(-\alpha t) - \exp(-\alpha T)}{a + 1 - \exp(-\alpha T)} \\
& \cdot \exp\left(\int_0^t \left(r(s) + \frac{1}{2}\|\xi(t)\|^2 \right) ds + \int_0^t \xi(s)^t dB_s \right)
\end{aligned}$$

As already computed: $X(0) = x$ ✓

We can see that the portfolio process $\pi^*(t)$ is equal to the one in example 3.13 whereas the capital process is different. However, the solution is quite general and complex.

We simplify our example by presuming that the market coefficients are constant on the interval $[0,T]$ (i.e. do not depend on time or chance).

- $\sigma := \sigma(t) \in \mathbb{R}^{n\times n}$, constant on $I = [0,T]$
- $b := b(t) \in \mathbb{R}^{n}$, constant on $I = [0,T]$
- $r := r(t) \in \mathbb{R}$, constant on $I = [0,T]$

Is this assumption realistic? For a short period, this assumption is certainly reasonable because the market coefficients will not change significantly during a brief interval. However, during a long period of time, it does not make sense to use constant coefficients as they might fluctuate highly. This in turn could influence the investor's decision of organizing his portfolio.

With these conditions we get the following solutions for the problem $(\check{P})$:

$$
\begin{aligned}
\xi &= \sigma^{-1}(b - re^n) \\
Q^* &= \frac{x\alpha}{\alpha + 1 - \exp(-\alpha T)} \cdot \exp\left(\int_0^T \left(r + \frac{1}{2}\|\xi\|^2 \right) ds + \int_0^T \xi^t dB_s \right) \\
&= \frac{\alpha x}{\alpha + 1 - \exp(-\alpha T)} \cdot \exp\left(rT + \frac{1}{2}\|\xi\|^2 T + \xi^t \int_0^T 1 dB_s \right) \\
&= \frac{\alpha x}{\alpha + 1 - \exp(-\alpha T)} \cdot \exp\left(rT + \frac{1}{2}\|\xi\|^2 T + \xi^t B_T \right) \\
C_t^* &= \frac{\alpha x \exp(-\alpha t)}{\alpha + 1 - \exp(-\alpha T)} \cdot \exp\left(rt + \frac{1}{2}\|\xi\|^2 t + \xi^t B_t \right) \\
X(t) &= x \cdot \frac{\alpha + \exp(-\alpha t) - \exp(-\alpha T)}{\alpha + 1 - \exp(-\alpha T)} \cdot \exp\left(rt + \frac{1}{2}\|\xi\|^2 t + \xi^t B_t \right) \\
\Rightarrow \quad C_t^* &= \frac{\alpha \exp(-\alpha t)}{1 + \exp(-\alpha t) - \exp(-\alpha T)} \cdot X(t) \\
\Rightarrow \quad \pi^* &= \left(\sigma^{-1}\right)^t \sigma^{-1}(b - re^n) \\
\Rightarrow \quad \vartheta_i^*(t) &= \frac{X(t)}{P_i(t)} \cdot \left(\left(\sigma^{-1}\right)^t \sigma^{-1}(b - re^n) \right)_i \qquad \text{for } i = 1, \dots, n \\
\Rightarrow \quad \vartheta_0^*(t) &= \frac{X(t)}{P_0(t)} \cdot \left(1 - \sum_{i=1}^n \left(\left(\sigma^{-1}\right)^t \cdot \xi \right)_i \right)
\end{aligned}
$$

4.2 Data and implementation

We want to apply the above calculated and specified solution to a portfolio which is composed of the 30 corporations of the main German stock index DAX.

Considering the solution of the problem in section 4.1, we need the following (constant) coefficients for every single company:

- the volatility (i.e the standard deviation)
- the expected return (i.e. the mean)
- the stock prices
- the market capitalization (see appendix A.5, will be used in section 4.3)

Moreover, we require the interest rate r. To get this information we break the former performances of the assets down and assume that the future expectation is contained in this analysis.

Via the internet (source: www.handelsblatt.com) we receive the closing prices of the 30 DAX securities from 08/26/2003 to 04/28/2006 (Hypo Real Estate 10/10/2003 - 04/28/2006). They were logarithmized and the daily differences of $\ln(P_i(t_i)) - \ln(P_i(t_{i-1}))$ were build.

With this data we are able to determine the expected returns and volatilities for one day. Having in mind that a stock exchange year has 200 trading days we obtain the yearly expected return and volatility for the corporations (compare stockprice.xls and table A.1 in appendix A.5).

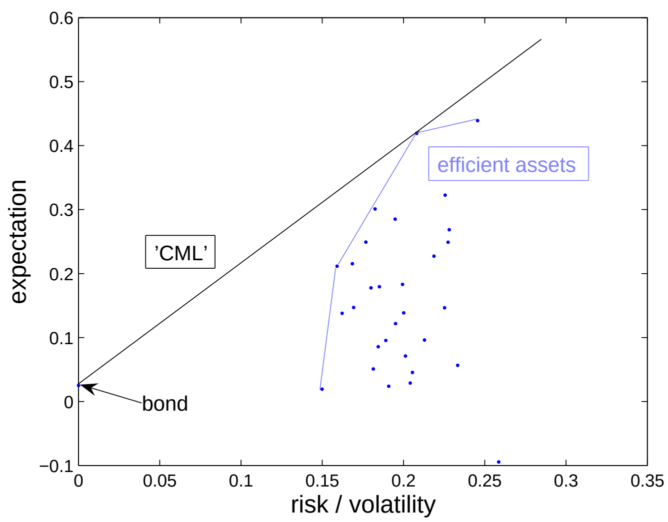

Figure 4.1: Efficient stocks and a variation of the CML for the DAX

The interest rate r has also been taken over by www.handelsblatt.com ($r = 0,025$). As we will vary this rate during the numerical tests, it is not necessary to analyze this value as exact as the stock prices. The figure shows efficient assets of the DAX or rather the combination of assets according to Markowitz [blue] and a 'variation' of the capital market line ['CML', black]. It is a variation because we only regard efficient stocks not efficient portfolios.

However, the computation of the expression $\sigma_{i\cdot} \cdot dB$, for $i = 1, \ldots, n$ turns out to be a little bit more difficult. From the existing data series we compute the correlation matrix, use the MATLAB command 'chol' and multiply the generated matrix with the Brownian motion dB. This procedure delivers the coherence among the 30 different Brownian motions and consequently the price development of the securities. For every company, $i = 1, \ldots, 30$, this expression has to be multiplied with the standard deviation of the asset.

Concerning the implementation of algorithm 3.15 in MATLAB it has to be pointed out that it is essential to clear all variables after every single flow of the main loop. Otherwise, the results are badly influenced: we obtain a negative capital value (a debt position), for instance. The expectation of the objective function Ψ is implemented with help of the main loop (compare file DAXconsumption.m). The Cholesky decomposition of the correlation matrix is smoothly executed by MATLAB. Other significant problems do not occur.

Numerical tests

We apply three different strategies to the portfolio problem $(\check{P})$:

1 trading and consumption process according to the martingale approach

2 a constant trading strategy with consumption in which $\frac{1}{31}$ of the *starting* capital is invested in every asset (including the bond)

3 a trading strategy with consumption in which $\frac{1}{31}$ of the *actual* capital value is invested in every asset (including the bond).

The first result is that strategy 2 is mostly superior to strategy 3 if they have the same consumption process. This is independent of the interest rate. Considering that 3 has a lower maximizer than 2 and that transaction costs accrue for 3 favors strategy 2.

Comparing strategies 1 - 3 with equal consumption processes (namely the one according to 1) we see that the martingale approach always provides a greater expected utility (i.e. a greater maximizer). This fact is not influenced by the choice of α and r.

If we want to analyze different consumption processes, the results eminently depend on the variables α and r and the consumption itself, needless to say. We take the consumption calculated by 1 compared to a percentage consumption of the actual capital ($C(t) = a \cdot X(t)$, $a \in (0, 1)$), for instance.

We get the following slight coherence:

1. the lower r, the greater the expected utility for 1
2. the higher a, the lower the expected utility for 2 and 3

One implication is significant here: The dependence of the utility for 1 on α is essential. Even small modifications on α have a deep impact on the maximizer. The higher we choose α, the higher the expected utility is.

Concerning the time it is obvious that

1. the longer the period, the higher the expected utility of 1 - 3
2. the smaller the increments, the smaller maximizers of 1 - 3

Furthermore, it is relevant that the dependence of utility on time is higher for the martingale approach than for the other strategies (compare also NumericalResults.xls).

Summary

- $\hookrightarrow$ the martingale approach is substantially dependent on time and the choice of α
- $\hookrightarrow$ the martingale approach is always superior to strategies 2 and 3 (with the same consumption process)

4.3 A comparison

To get a picture of the effectiveness of the martingale approach, we apply two different trading strategies to our familiar portfolio of the 30 DAX corporations and compare them to the presented solution.

The shares of the stocks and the bond

- A are calculated according to the martingale approach (i.e. not constant)
- B are set equal (i.e. $\frac{1}{31}$ of the starting capital is invested in every asset including the bond $\rightarrow$ constant)
- C are computed according to the market capitalization of the companies (i.e. constant, no share of the bond)

One additional opportunity is to adapt B and C in every step, i.e. the constant share of the actual capital value (not of the starting capital) is invested. This implies that our trading strategy is not constant anymore and that transaction costs incur. However, numerical tests show that these 'adapting' strategies lead to a smaller maximizer than the constant ones, the costs for transaction still not included. Hence, we consider B and C.

Again, we choose the logarithmic utility and assume that the market coefficients are constant. In contrast to the previous settings we suppose that the investor is not interested in consumption.

This leads to the problem $(\check{P}_2)$:

$$(\check{P}_2) \begin{cases} \max \; E\left[\ln(X^{x,\pi}(T))\right] \\ \text{s.t. } (\pi, 0) \in D^*_\pi(x) \end{cases}$$

Applying 3.15 we receive:

$$\begin{aligned}
& U_2(x) = \ln(x) \quad \Rightarrow \quad U_2'(x) = \frac{1}{x} \quad \Rightarrow \quad \hat{x} = \infty \\
\Rightarrow \quad & I_2(y) = \frac{1}{y} \quad \Rightarrow \quad A(\lambda) = \frac{1}{\lambda} \quad \Rightarrow \quad A^{-1}(x) = \frac{1}{x} \\
\Rightarrow \quad & Q^* = \frac{x}{\delta(T)} \\
\Rightarrow \quad & X(t) = x \cdot \exp\left(\left(r + \frac{1}{2}\|\xi\|^2 \right) t + \xi^t dB_t \right) := f(t, B_1, \dots, B_n) \\
\Rightarrow \quad & \nabla_B f(t, B_1, \dots, B_n) = X(t) \cdot \xi^t \\
\Rightarrow \quad & \pi^*(t) = \left(\sigma^{-1}\right)^t \cdot \xi
\end{aligned}$$

Hence, the different trading strategies arise as

A

$$\vartheta_i^*(t) = \frac{X(t)}{P_i(t)} \left(\sigma^{-1}\right)^t \cdot \xi \qquad \text{for the 30 stocks}$$

$$\vartheta_0^*(t) = \frac{X(t)}{P_0(t)} \left(1 - \sum_{i=1}^{3} 0 \left(\left(\sigma^{-1}\right)^t \cdot \xi \right)_i \right) \qquad \text{for the bond}$$

B

$$\vartheta_i \equiv \frac{1}{31} \qquad \text{for } i = 0, \dots, 30$$

C

$$\vartheta_i \equiv \frac{\text{market capitalization of asset } i}{\text{market capitalization DAX}} \qquad i = 1, \dots, 30$$

Numerical tests

The basic and significant result of the implementation (compare file DAXcomparison.m) is that the martingale approach always delivers a higher utility of the terminal wealth compared to the constant strategies. Under any condition (modification of the length of the period $(= T)$, the increments $(= dt)$, the starting capital $(= x)$ or the interest rate $(= r)$) strategy A provides a higher maximizer.

The influence of the starting capital and the interest rate on A to C is weak, whereas the impact of modifications concerning the time (T and dt) is strong. We sum up some observations (compare NumericalResults.xls):

- the longer the period, the higher the utility for A - C
- the smaller the increments, the higher the utility for A - C
- a comparison of strategy B and C is quite difficult: which one is superior depends heavily on the underlying conditions; in most cases B yields a higher terminal value as c

The maximizer computed via the martingale approach exceeds those of the constant strategies. In economic terms: the martingale approach creates value-added to our portfolio compared to a passive portfolio management (B, C).

5 Conclusion

The martingale approach delivers one answer to the question of how to optimize a portfolio. A great advantage of the presented solution is that it is continuous in time: the investor is able to consume or to redistribute his portfolio at any time instance during his investment. Furthermore, the measurability of utility (or expected utility) of consumption and terminal value is relinquished to the investor by the free choice of the utility functions; a helpful component to manage his portfolio.

The main property of the martingale approach is obvious: it requires a complete market. Without this assumption its derivation (the decomposition of the portfolio problem (P) into the static problem (S) and the representation problem (R)) and the resulting solutions (theorems 3.10 and 3.14) would not work out.

The simple form of the solution of the representation problem (R) is based on the constraint of constant market coefficients. This implies that the investment period should not be too long as the coefficients otherwise might fluctuate. Unfortunately, numerical tests show that the longer the period the higher the expected utility is.

Comparing different consumption processes stays difficult because it depends on the chosen utility functions.

We observe that the shares to hold (compare 'theta' in the MATLAB files) are strongly dependent on the interest rate $(= r)$; an evidentiary consequence.

However, the martingale approach often recommends short sellings and credits. In reality this aspect results in an augmentation of risk for the investor. Vice versa, the taking advantage of short sellings and credits leads to high value-added compared to a passive portfolio management.

<u>An outlook</u>

The numerical studies show the high effectiveness of the martingale approach compared to passive strategies. But: the costs of transaction should be considered to get a <u>real</u> result.

As our solution proposes to redistribute the portfolio in every step, the costs of transaction will badly influence the capital value.

Thus, if we included these costs, we will discover whether the martingale approach is really superior to a passive portfolio management and whether it might be a useful application for the practise.

A Mathematical tools

A.1 Probability spaces

For the introduction into the complex mathematical tools for modelling the financial market, we recall some basic definitions and notations of the theory of probability.
Let Ω be a given set and $\mathcal{F} \subseteq \Omega$.

A.1 Definition

a) $\mathcal{F}$ is a ***σ-algebra***, if $\mathcal{F}$ holds the following properties:

   i) $\emptyset \in \mathcal{F}$

   ii) $F \in \mathcal{F} \Rightarrow F^c := \Omega \setminus F \in \mathcal{F}$

   iii) $F_1, F_2, \ldots \in \mathcal{F} \Rightarrow F := \bigcup_{n=1}^{\infty} F_n \in \mathcal{F}$

b) A pair $(\Omega, \mathcal{F})$ with a σ-algebra $\mathcal{F}$ is called ***measurable space*** .

c) A function $P : \mathcal{F} \to [0,1]$ on a measurable space $(\Omega, \mathcal{F})$ such that

   i) $P(\emptyset) = 0$

   ii) $P(\Omega) = 1$

   iii) $F_1, F_2, \ldots \in \mathcal{F}, F_i \cap F_j = \emptyset \; \forall \, i, j, \, i \neq j$ with
   $P\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} P(F_n)$

   is called ***probability measure*** (i.e. $P(F)$ = 'probability that F occurs').

Notation

A measurable space $(\Omega, \mathcal{F})$ connected with probability measure P is known as ***probability space*** $(\Omega, \mathcal{F}, P)$. $(\Omega, \mathcal{F}, P)$ is called ***complete probability space***, if $\mathcal{F}$ contains all subsets $G \subset \Omega$ with P - outer measure zero
(i.e. $P(G) := \inf\{P(F), F \in \mathcal{F}, G \subset F\} = 0$).

A.2 Definition

Let $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ be measurable spaces and $X : \Omega \to \Omega'$ a function.

i) X is ***$\mathcal{F}$-$\mathcal{F}'$-measurable***, if it holds that $X^{-1}(F') \in \mathcal{F}$ for all $F' \in \mathcal{F}'$.

   $X^{-1}(F') := \{\omega \in \Omega : X(\omega) \in F'\}$

   If $(\Omega', \mathcal{F}') = (\mathbb{R}^n, \mathcal{B}^n)$ X is ***$\mathcal{F}$-measurable***, n $\in \mathbb{N}$.
   $\mathcal{B}^n$:= Borel-σ-algebra on $\mathbb{R}^n$, $\sigma(\mathbb{R}^n)$, smallest σ-algebra which contains all open subsets of $\mathbb{R}^n$.

ii) A ***random variable*** X is a $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$.

A.3 Definition
Let $X : \Omega \to \mathbb{R}$ be a random variable on a probability space $(\Omega, \mathcal{F}, P)$.

i) The ***distribution*** μ_X of X is defined as
$\mu_X(x) := P(X \leqq x) = P(\omega \in \Omega \mid X(\omega) \leqq x) :\Leftrightarrow \mu_X(B) = P\left(X^{-1}(B)\right)$

ii) If a function $\nu_X : \mathbb{R} \to \mathbb{R}$ exists, such that
$\mu_X(x) = \int\limits_{-\infty}^{x} \nu_X(s)ds :\Leftrightarrow \mu_X(B) = \int\limits_{B} \nu_X(x)dx$ for $B \in \Omega$ is held,
we call ν_X ***density*** of X.

iii) The ***expectation*** $E[X]$ of X is defined as

a) $E[X] := \int\limits_{\Omega} X(\omega)dP(\omega) = \int\limits_{\mathbb{R}^n} x d\mu_X(x) = \int\limits_{\mathbb{R}^n} x\nu_X dx$

b) $f : \mathbb{R}^n \to \mathbb{R}$ $E[f(x)] := \int\limits_{\Omega} f(X(\omega))dP(\omega) = \int\limits_{\mathbb{R}^n} f(x)\nu_X(x)dx$

iv) The ***variance*** $Var(X)$ of X is
$Var(X) := E\left[(X - E[X])^2\right] = E\left[X^2\right] - E[X]^2$

v) $\sigma := +\sqrt{Var(X)}$ is called ***standard deviation*** of X.

A.2 Stochastic tools and probability beliefs

Some important theorems are used in this thesis. We just present them without proof.

A.4 Theorem (Dominated convergence)
Let X_n, X be $\mathcal{F}$-measurable, $n \in \mathbb{N}$, $X_n \to X$ P-a.e. and
$|X_n| \leq Y$ for P-a.e. $n \in \mathbb{N}$ with $Y \in \mathcal{L}^2[0,T]$

i) $\Rightarrow$ $X \in \mathcal{L}^2[0,T]$

ii) $\Rightarrow$ $\int XdP = \lim\limits_{n\in\mathbb{N}} \int X_n dP = \int \lim\limits_{n\in\mathbb{N}} X_n dP$

Without proof.

A.5 Theorem (Monotone convergence)
Let X_n, $X \in \mathcal{L}^2[0,T]$, $n \in \mathbb{N}$, satisfy $X_n \nearrow X$.

$\Rightarrow$ $\sup\limits_{n\in\mathbb{N}} X_n \in \mathcal{L}^2[0,T]$

$\Rightarrow$ $\int \sup\limits_{n\in\mathbb{N}} X_n dP = \sup\limits_{n\in\mathbb{N}} \int X_n dP$

Without proof.

A.6 Theorem (Lemma of Fatou)
Let $X_n \in \mathcal{L}^2[0,T]$, $n \in \mathbb{N}$, satisfy $X_n \geq 0$.

$$\Rightarrow \quad \int \liminf_{n\to\infty} X_n \leq \liminf_{n\to\infty} \int X_n dP$$

Without proof.

Excursion

We remember the main theorem of the differential- and integral-calculus:

A.7 Theorem
Let $f \in \mathcal{C}^1$ on $[a,b] \in \mathbb{R} \quad \Rightarrow \quad \int\limits_a^b f'(x)dx = f(b) - f(a)$

Without proof.

A.8 Theorem (Multi-dimensional Itô formula)
Let $X(t,\omega) = (X_1(t,\omega),\ldots,X_n(t,\omega))^t$ be a n-dimensional Itô process with

$$X_i(t,\omega) = X_i(0,\omega) + \int\limits_0^t K_i(s,\omega)ds + \sum_{j=1}^m \int\limits_0^t L_{ij}(s,\omega)dB_j(s,\omega) \quad \text{for } i = 1,\ldots,n$$

in which $B(t,\omega) = (B_1(t,\omega),\ldots,B_m(t,\omega))^t$ is a m-dimensional Brownian motion.

Assume $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, $f \in \mathcal{C}^{1,2}$. Then we have:

$$\begin{aligned}
&f(t,X(t,\omega)) = f(t,X_1(t,\omega),\ldots,X_n(t,\omega)) \\
=&f(0,X_1(0,\omega),\ldots,X_n(0,\omega)) \\
+&\int\limits_0^t \frac{\partial f}{\partial t}\left(s,X_1(s,\omega),\ldots,X_n(s,\omega)\right) ds \\
+&\sum_{i=1}^n \int\limits_0^t \frac{\partial f}{\partial x_i}\left(s,X_1(s,\omega),\ldots,X_n(s,\omega)\right) dX_i(s,\omega) \\
+&\frac{1}{2}\sum_{j=1}^m \sum_{i=1}^n \int\limits_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}\left(s,X_1(s,\omega),\ldots,X_n(s,\omega)\right) d\langle X_i,X_j\rangle(s,\omega)
\end{aligned} \tag{A.1}$$

Proof : see Korn [4], pp. 59-60.
The proof is similar to the one of the one-dimensional Itô formula.

A.3 Properties of utility functions

In chapter 3.1 we introduce utility functions. Some properties of these functions are elementary components of the depiction of the martingale approach (see chapter 3.2 $\rightsquigarrow$ 'A heuristic optimization'). That is why we summarize them in this section.

Assumptions: u and U are utility functions in sense of definition 3.1.

A.9 Definition
Let be $x, y \in (0, \infty)$.

$$u(x) \text{ is } \left\{ \begin{array}{c} \textbf{\textit{strictly concave}} \\ \textbf{\textit{concave}} \\ \textbf{\textit{strictly convex}} \\ \textbf{\textit{convex}} \end{array} \right\} :\Leftrightarrow \; u'(x) \left\{ \begin{array}{c} > \\ \geq \\ < \\ \leq \end{array} \right\} \tfrac{u(x)-u(y)}{x-y}$$

$U_t(x) := U(t, x)$ for any fixed $t \in I$, $U'_t(x) := U'(t, x) := \frac{\partial U}{\partial x}(t, x)$.
Remember that $U'_t \in \mathcal{C}^0$. Hence, we get

$$U_t(x) \text{ is } \left\{ \begin{array}{c} \textbf{\textit{strictly concave}} \\ \textbf{\textit{concave}} \\ \textbf{\textit{strictly convex}} \\ \textbf{\textit{convex}} \end{array} \right\} :\Leftrightarrow \; U'_t(x) \left\{ \begin{array}{c} > \\ \geq \\ < \\ \leq \end{array} \right\} \tfrac{U_t(x)-U_t(y)}{x-y}$$

A.10 Theorem (Properties of utility functions)
Let u, U be utility functions. This implies

1. u, U are strictly increasing
2. u', U'_t are strictly decreasing
3. u', $U'_t \in \mathcal{C}^0$, $u' : (0, \infty) \to (0, \infty)$, $U'_t : [0, T] \times (0, \infty) \to (0, \infty)$
4. $\exists!\ i := (u')^{-1}$ with
   i) $i : (0, \infty) \to (0, \infty)$
   ii) i is strictly decreasing and $i \in \mathcal{C}^0$
5. $\exists!\ I_t := (U'_t)^{-1}$ with
   i) I_t is strictly decreasing and $I_t \in \mathcal{C}^0$
6. $u(i(y)) \geq u(x) + y(i(y) - x)$
7. $U_t(I_t(y)) \geq U_t(x) + y(I_t(y) - x)$
8. $\lim_{y \downarrow 0} i(y) = +\infty$, $\lim_{y \to \infty} i(y) = 0$
9. $\lim_{y \downarrow 0} I_t(y) = +\infty$, $\lim_{y \to \infty} I_t(y) = 0$

Sketch of proof
For 1) to 3) use the assumptions in definition 3.1.
4) and 5): u' is strictly decreasing and $u' \in \mathcal{C}$.
6) and 7): Use the concavity of u and U.
8) and 9): Follows from 4) and 5).

A.4 KKT-conditions and Lagrangian multipliers

In chapter 3.2 we derive the solution for the portfolio problem (P) with help of the Lagrangian multiplier method often used for the non-linear programming. Here, we just want to present the main results and implications of this optimization theory adjusted to our heuristic solution.
We regard the following problem (NLP):

$$(NLP)\begin{cases} \max\ \Psi(Q) \\ \text{s.t. } g(Q) \leq 0 \end{cases}$$

in which $\Psi,\ g : \mathbb{R} \to \mathbb{R}$, $\Psi,\ g \in \mathcal{C}^1$ and Ψ is strictly concave, g is convex.

A.11 Definition

i) The function $L : \mathbb{R}^2 \to \mathbb{R}$ with

$$L(Q, \lambda) := \Psi(Q) - \lambda \cdot g(Q)$$

is called ***Lagrange function of*** (NLP)

ii) The ***Karush-Kuhn-Tucker-conditions (KKT-conditions)*** are

(a)
$$g(Q) \leq 0 \tag{A.2}$$

(b)
$$\Psi'(Q) - \lambda \cdot g'(Q) = 0 \tag{A.3}$$

(c)
$$\lambda \geq 0,\ \lambda \cdot g(Q) = 0 \tag{A.4}$$

A.12 Theorem

i) If a solution Q^* for (NLP) exists, Q^* is optimal.

ii) Let be Q^* a solution for (NLP), then the KKT-conditions are valid.

iii) If the Lagrangian multiplier holds $\lambda > 0$ for Q^*, Q^* even solves the problem $(\overline{NLP})$:

$$(\overline{NLP})\begin{cases} \max\ \Psi(Q) \\ \text{s.t. } g(Q) = 0 \end{cases}$$

Sketch of proof
i) As Ψ is strictly concave, Q^* is maximizer.
ii) Without proof.
iii) With (A.4) and $\lambda > 0$ follows $g(Q^*) = 0$.

A.5 The DAX corporations

The key-data are derived from the closing prices from 08/26/2003 to 04/28/2006 (Hypo Real Estate 10/10/2003 to 04/28/2006), the source was www.handelsblatt.com.

Table A.1: The 30 DAX corporations, April 28, 2006

ASSET	CLOSING PRICE	MEAN	DEVIATION
adidas	167.50	0.2492	0.1768
Allianz	132.55	0.1386	0.2001
Altana	50.78	0.0289	0.2041
BASF	67.83	0.1379	0.1622
Bayer	36.51	0.1831	0.1994
BMW	43.11	0.0510	0.1814
Commerzbank	32.75	0.2684	0.2281
Continental	94.40	0.4191	0.2081
DaimlerChrysler	43.65	0.0711	0.2012
Deutsche Bank	97.04	0.1777	0.1800
Deutsche Börse	114.73	0.2852	0.1948
Deutsche Post	21.15	0.1219	0.1951
Deutsche Telekom	14.36	0.0193	0.1499
EON	97.00	0.2116	0.1591
Fresenius	95.05	0.2154	0.1685
Henkel	95.26	0.1469	0.1693
Hypo	55.29	0.4390	0.2456
Infineon	9.63	-0.0945	0.2586
Linde	71.02	0.1795	0.1852
Lufthansa	14.59	0.0454	0.2055
MAN	60.05	0.3225	0.2256
Metro	45.11	0.0955	0.1891
Münchener Rück	112.20	0.0239	0.1909
RWE	68.72	0.3011	0.1825
SAP	173.50	0.1466	0.2253
Schering	85.07	0.2490	0.2274
Siemens	75.21	0.0856	0.1844
ThyssenKrupp	26.09	0.2272	0.2187
TUI	16.89	0.0565	0.2333
VW	61.45	0.0962	0.2129

The next table shows the market capitalization of the 30 DAX corporations (according to www.onvista.de).

Table A.2: Market capitalization in millions of EURO, May 5, 2006

ASSET	MARKET CAPITALIZATION
adidas	8352.80
Allianz	54588.02
Altana	6902.06
BASF	35024.01
Bayer	26014.78
BMW	26556.69
Commerzbank	20912.91
Continental	13491.03
DaimlerChrysler	43913.79
Deutsche Bank	50527.36
Deutsche Börse	11793.37
Deutsche Post	6731.42
Deutsche Telekom	25999.42
EON	60624.20
Fresenius	64868.08
Henkel	9114.09
Hypo	5656.11
Infineon	7288.16
Linde	7381.46
Lufthansa	8414.46
MAN	9969.31
Metro	14772.91
Münchener Rück	25931.09
RWE	35696.22
SAP	56051.01
Schering	16545.84
Siemens	65958.15
ThyssenKrupp	14431.42
TUI	4214.36
VW	16548.08

B Proofs

The proofs presented in this chapter follow the foregoing of Korn (compare Korn [2, 4]). Some parts are left out and others are more detailed.

Proof of theorem 1.13, chapter 1.2

W.l.o.g. let be $\phi := \phi^{(p)}$, $t := t_k$, $k < p$, $k, p \in \mathbb{N}$. Then

$$\begin{aligned}
E\left[(I_t[\phi])^2\right] &= E\left[\left(\int_0^t \phi_s dB_s\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{p} \varphi_i [B_{t_i \wedge t} - B_{t_{i-1} \wedge t}]\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{k} \varphi_i [B_{t_i \wedge t} - B_{t_{i-1} \wedge t}]\right)^2\right] \quad (\curvearrowright t := t_k) \\
&= E\left[\left(\sum_{i=1}^{k} \varphi_i [B_{t_i} - B_{t_{i-1}}]\right)^2\right] \quad (\curvearrowright \text{complete induction}) \\
&= E\left[\sum_{i=1}^{k}\sum_{j=1}^{k} \varphi_i \varphi_j [B_{t_i} - B_{t_{i-1}}][B_{t_j} - B_{t_{j-1}}]\right]
\end{aligned}$$

For $i \neq j$, w.l.o.g. $i > j$, we have

$$\begin{aligned}
&E\left[\varphi_i \varphi_j [B_{t_i} - B_{t_{i-1}}][B_{t_j} - B_{t_{j-1}}]\right] \\
=&E\left[\varphi_i [B_{t_i} - B_{t_{i-1}}] \varphi_j [B_{t_j} - B_{t_{j-1}}]\right] \quad (\curvearrowright \text{theorem 1.5 i.}) \\
=&E\left[\varphi_i [B_{t_i} - B_{t_{i-1}}] | \mathcal{F}_{t_{i-1}}\right] E\left[\varphi_j [B_{t_j} - B_{t_{j-1}}]\right] \\
=&\varphi_i E\left[B_{t_i} - B_{t_{i-1}} | \mathcal{F}_{t_{i-1}}\right] E\left[\varphi_j [B_{t_j} - B_{t_{j-1}}]\right] \quad (\curvearrowright \text{theorem 1.5 ii.}) \\
=&\varphi_i E\left[B_{t_i} - B_{t_{i-1}}\right] E\left[\varphi_j [B_{t_j} - B_{t_{j-1}}]\right] \\
=&0 \quad (\text{as } B_{t_i} - B_{t_{i-1}} \sim N(0, t_i - t_{i-1}))
\end{aligned}$$

For $i = j$, we have

$$\begin{aligned}
E\left[(\varphi_i)^2 [B_{t_i} - B_{t_{i-1}}]^2\right] =& E\left[(\varphi_i)^2 E[(B_{t_i} - B_{t_{i-1}})^2]\right] \quad \curvearrowright (\text{theorem 1.5 i.}) \\
=& E\left[(\varphi_i)^2 Var(B_{t_i} - B_{t_{i-1}})\right] \\
=& E\left[(\varphi_i)^2 (t_i - t_{i-1})\right] \\
\Rightarrow \quad E\left[(I_t[\phi])^2\right] =& E\left[\sum_{i=1}^{k} \varphi_i^2 (t_i - t_{i-1})\right] \\
(\text{definition 1.11}) =& E\left[\int_0^t \left(\phi_s^{(p)}\right)^2 ds\right] \quad \blacksquare
\end{aligned}$$

Proof of theorem 1.14, chapter 1.2

We prove theorem 1.14 in three steps.

i) Let $X \in \mathcal{L}^2[0,T]$ be bounded and continuous $\forall\ \omega \in \Omega$ (i.e. all paths $X_t(\cdot)$ are continuous). We define

$$\phi_t^{(n)}(\omega) := X_0(\omega)\chi_{\{0\}}(t) + \sum_{i=1}^{2^n-1} X_{\frac{iT}{2^n}}(\omega) \cdot \chi_{(\frac{iT}{2^n}, \frac{(i+1)T}{2^n}]}(t)$$
$$\Rightarrow \phi_t^{(n)} \in \mathcal{S} \qquad \text{(Definition 1.9)}$$
$$\Rightarrow \phi_t^{(n)} \in \mathcal{L}^2[0,T] \qquad \text{(because } X_t \in \mathcal{L}^2[0,T])$$

We see that

$$(+) \quad \lim_{n\in\mathbb{N}} \phi_t^{(n)} = X_t \qquad P\text{-a.s. since } X_t(\cdot) \text{ is continuous } \forall\ \omega \in \Omega.$$

Moreover, we notify that

- $X_t \geq 0 \Rightarrow \phi_t^{(n)} \geq 0$
- $X_t, \phi_t^{(n)}$ are $\mathcal{F}_t$-measurable $\forall\ t \in [0,T]$

Define $Y_t := \underbrace{(X_t - \phi_t^{(n)})^2}_{:=\rho}$.

$\Rightarrow Y_t \in \mathcal{L}^2[0,T]$
$\Rightarrow Y_t \geq |\rho|$

Consequently, we can apply the theorem of dominated convergence (see theorem A.4, appendix A.2):

$$\lim_{n\in\mathbb{N}} E\left[\int_0^T \left(X_t - \phi_t^{(n)}\right)^2 dt\right] = \lim_{n\in\mathbb{N}} \int_\Omega \int_0^T \left(X_t - \phi_t^{(n)}\right)^2 dtdP$$
$$= \int_\Omega \int_0^T \lim_{n\in\mathbb{N}} \left(X_t - \phi_t^{(n)}\right)^2 dtdP$$
$$\stackrel{(+)}{=} 0$$

ii) Let $X_t \in \mathcal{L}^2[0,T]$ be bounded and

a) $Z_t(\omega) := \int_0^t X_s(\omega)ds$

b) $\tilde{X}_t^m(\omega) := \frac{Z_t(\omega) - Z_{(t-\frac{1}{m})}(\omega)}{\frac{1}{m}}$, $m \in \mathbb{N}$, $m \gg 0$ such that

- $t - \frac{1}{m} \geq 0$
- Z_t continuous, bounded and $\mathcal{F}_t$-measurable.

The main theorem of the differential- and integral-calculus (theorem A.7, appendix A.2) proves that

$$\begin{aligned}
\lim_{m\to\infty} \tilde{X}_t^m(\omega) &= \lim_{m\to\infty} \frac{Z_t(\omega) - Z_{(t-1/m)}(\omega)}{1/m} \\
&= \lim_{m\to\infty} \frac{\int_0^t X_s(\omega)ds - \int_0^{t-1/m} X_s(\omega)ds}{1/m} \\
&= X_t(\omega) \qquad \text{for } \lambda \otimes P\text{-a.e. } (t,\omega) \in [0,T] \times \Omega \\
&\Rightarrow \quad \tilde{X}_t^m(\omega) \in \mathcal{L}^2[0,T]
\end{aligned}$$

and with the theorem of dominated convergence follows

$$\Rightarrow E\left[\int_0^T \left(\tilde{X}_t^m - X_t\right)^2 dt\right] \to 0 \text{ as } m \to \infty$$

We produce a sequence $\{\phi^{(n)}\}$ of elementary processes with $\phi_t^{(n)} := \tilde{X}_t^{m_n}$ as in i) and we have

$$\lim_{n\in\mathbb{N}} E\left[\int_0^T \left(\phi_t^{(n)} - X_t\right)^2 dt\right] = 0$$

iii) Let $X \in \mathcal{L}^2[0,T]$. Define $\tilde{X}_t^m(\omega) := X_t(\omega) \cdot \chi_{\{\omega\in\Omega | X_t(\omega) \le m\}}(\omega)$
$\Rightarrow \quad |\tilde{X}_t^m(\omega)| \le \infty$
$\Rightarrow \quad |\tilde{X}_t^m(\omega)| \le |X_t(\omega)|$
Hence, we can apply the theorem of dominated convergence again and obtain

$$E\left[\int_0^T \left|\tilde{X}^n - X_t\right|^2 dt\right] \to 0 \qquad \text{as } m \to \infty$$

With i) and ii) we get for $\phi_t^{(n)} := \tilde{X}_t^{m_n} \in \mathcal{S}$:

$$\lim_{n\in\mathbb{N}} E\left[\int_0^T \left(\phi_t^{(n)} - X_t\right)^2 dt\right] = 0$$

∎

Proof of lemma 2.1, chapter 2.1

i) W.l.o.g. let be $m = 1$:

$$\begin{aligned} E\left[P_i(t)\right] &= E\left[p_i \cdot \exp\left(b_i \cdot t - \frac{1}{2}\sigma_i^2 \cdot t + \sigma_i B(t)\right)\right] \\ &= p_i \cdot \exp(b_i \cdot t) E\left[\exp\left(-\frac{1}{2}\sigma_i^2 \cdot t + \sigma_i B(t)\right)\right] \quad (\curvearrowright B_t \sim N(0,t)) \\ &= p_i \cdot \exp(b_i \cdot t) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\sigma_i^2 \cdot t + \sigma_i \cdot x\right) \cdot \frac{1}{\sqrt{t2\pi}} \exp\left(-\frac{x^2}{2t}\right) dx \\ &= p_i \cdot \exp(b_i \cdot t) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x - \sigma_i \cdot t)^2}{2t}\right) dx}_{= \text{density for a random variable} \sim N(\sigma_i \cdot t, t) = 1} \\ &= p_i \cdot \exp(b_i \cdot t) \end{aligned}$$

ii) W.l.o.g. again $m = 1$. Use $Var(X) = E\left[X^2\right] - E\left[X\right]^2$ and go on as in i).

iii) W.l.o.g. $m = 1$. We show that $E[Y_t|\mathcal{F}_s] = Y_s$

$$\begin{aligned} E\left[Y_t|\mathcal{F}_s\right] &= E\left[a \cdot \exp\left(c_1 B_t - \frac{1}{2}c_1^2 \cdot t\right) |\mathcal{F}_s\right] \\ &= \underbrace{a \cdot \exp\left(c_1 B_s - \frac{1}{2}c_1^2 \cdot s\right)}_{=Y_s} \\ &\quad \underbrace{\cdot E\left[\exp\left(c_1(B_t - B_s) - \frac{1}{2}c_1^2(t-s)\right) |\mathcal{F}_s\right]}_{B_t - B_s \text{ independent of } \mathcal{F}_s} \\ &= Y_s \cdot \underbrace{E\left[\exp\left(c_1(B_t - B_s) - \frac{1}{2}c_1^2(t-s)\right)\right]}_{=1 \text{ see i}} = Y_s \end{aligned}$$

■

Proof of theorem 2.7, chapter 2.3

i) Note that $\delta(t) > 0, C(t) \geq 0$ and with $\pi(t) \in D_\pi(x)$

$\Rightarrow \quad X_t \geq 0$

$\Rightarrow \quad 0 \geq E\left[\delta(t)X_t + \int_0^t \delta(s)C(s)ds\right].$

Further:

$\delta(0) = 1$, $X_0 = x$,

$$d\delta(t) = \overbrace{-\delta(t)r(t)}^{=K_t} dt \overbrace{-\delta(t)\xi(t)^t}^{L_t} dB_t$$

and

$$dX_t = \underbrace{\left(X_t r(t) - X_t C(t) + X_t \pi(t)^t (b(t) - r(t)e^n)\right)}_{N_t} dt + \underbrace{X_t \pi(t)^t \sigma(t)}_{O_t} dB_t$$

$\Rightarrow \quad \delta(t), X_t \in \mathcal{J}.$

We apply theorem 1.24, chapter 1.2:

$$\delta(t)X_t = x + \int_0^t \delta(s)X_s \Bigg[r(s) - \frac{C(s)}{X_s} + \pi(s)^t(b(s) - r(s)e^n) - r(s)$$
$$- \pi(s)^t\sigma(s)\xi(s) \Bigg] ds + \int_0^t \delta(s)X_s \left(\pi(s)^t\sigma(s) - \xi(s)^t\right) dB_s$$
$$= x + \int_0^t \delta(s)X_s \Bigg[-\frac{C(s)}{X_s} + \pi(s)^t(b(s) - r(s)e^n)$$
$$- \pi(s)^t(b(s) - r(s)e^n) \Bigg] ds + \int_0^t \delta(s)X_s \left(\pi(s)^t\sigma(s) - \xi(s)^t\right) dB_s$$

$$\Rightarrow \quad \underbrace{\delta(t)X_t + \int_0^t \delta(s)C(s)ds}_{\geq 0} = x + \int_0^t \delta(s)X_s \left(\pi(s)^t\sigma(s) - \xi(s)^t\right) dB_s \odot$$

Korn [4] shows that a non-negative local martingale is a supermartingale. Hence, we receive:

$$0 \leq E\left[\delta(t)X_t + \int_0^t \delta(s)C(s)ds\right]$$
$$= E\left[x + \int_0^t \delta(s)X_s \left(\pi(s)^t\sigma(s) - \xi(s)^t\right) dB_s\right]$$
$$(\curvearrowright \text{Korn [4], Satz 17}) \quad \leq x$$

ii) We define:

- $X_t := \delta(t)^{-1}E\left[\int_t^T \delta(s)C(s)ds + \delta(T)Q|\mathcal{F}_t\right]$
  Note that, due to the measurability of δ, C and Q, X_t is $\mathcal{F}_t$-measurable. Further, $X_t \geq 0$ because $\delta(t) > 0$, $C \geq 0$ and $B \geq 0$.
  We have $X_0 = E\left[\int_0^T \delta(s)C(s)ds + \delta(T)Q|\mathcal{F}_0\right]$.
  As $\mathcal{F}$ is the Brownian filtration, the conditional expectation w.r.t. $\mathcal{F}_0$ is constant, i.e.
  $X_0 = E\left[\int_0^T \delta(s)C(s)ds + \delta(T)Q\right] = x$ (see assumptions in ii)).
  $X_T = \delta(T)^{-1}E\left[\delta(T)Q|\mathcal{F}_T\right] = E\left[Q|\mathcal{F}_T\right] = Q$, because Q is $\mathcal{F}_T$-measurable (compare theorem 1.5 ii), chapter 1.1).

- $M_t := E\left[\int\limits_0^T \delta(s)C(s)ds + \delta(T)Q|\mathcal{F}_t\right]$

Thus, we get

$$\begin{aligned}\underline{X_t\delta(t) + \int\limits_0^t \delta(s)C(s)ds} &= E\left[\int\limits_t^T \delta(s)C(s)ds + \delta(T)Q|\mathcal{F}_t\right] + \int\limits_0^t \delta(s)C(s)ds \\ &= E\left[\int\limits_0^T \delta(s)C(s)ds + \delta(T)Q|\mathcal{F}_t\right] \\ &= \underline{M_t}\end{aligned}$$

and for $r \leq t$

$$\begin{aligned}E\left[M_t|\mathcal{F}_r\right] &= E\left[E\left[\int\limits_0^T \delta(s)C(s)ds + \delta(T)Q|\mathcal{F}_t\right]|\mathcal{F}_r\right] \\ \text{(theorem 1.5 i))} &= E\left[\int\limits_0^T \delta(s)C(s)ds + \delta(T)Q|\mathcal{F}_r\right] \\ &= M_r\end{aligned}$$

$\Rightarrow M_t$ is a Brownian martingale (with Brownian filtration $\mathcal{F}_t$).

$$M_t \text{ satisfies } E\left[M_t^2\right] = E\left[\underbrace{\left(E\left[\int\limits_0^T \delta(s)C(s)ds + \delta(T)Q|\mathcal{F}_t\right]\right)^2}_{\leq x^2}\right] < \infty$$

Consequently, we can apply the martingale representation theorem 1.25, chapter 1.2:

$\Rightarrow \exists$ a $\mathcal{F}_t$-adapted process κ_t with $P\left(\int\limits_0^T \|\kappa_s\|^2 ds < \infty\right) = 1$ and

$$\begin{aligned}M_t &= M_0 + \int\limits_0^t \kappa_s^t \cdot dB_s \qquad \forall\ t \in I,\ P\text{-a.s.} \\ M_0 &= X_0\delta(0) = x \\ M_t &= X_t\delta(t) + \int\limits_0^t \delta(s)C(s)ds \\ &\overset{\odot}{=} x + \int\limits_0^t X_s\delta(s)\left(\pi(s)^t\sigma(s) - \xi(s)\right)^t dB_s \\ \Rightarrow \kappa_s &= X_s\delta(s)\left(\pi(s)^t\sigma(s) - \xi(s)\right)\end{aligned}$$

for $X_s \neq 0$:

$$\frac{\kappa_s}{X_s\delta(s)} + \xi(s) = \pi(s)^t\sigma(s)$$

$$\Rightarrow \pi(t)^t = \begin{cases} (\sigma(t)^{-1})^t\left(\frac{\kappa_t}{X_t\delta(t)} + \xi(t)\right), & \text{for } X_t \neq 0; \\ 0, & \text{for } X_t = 0. \end{cases}$$

The last thing to prove is that $(\pi, C) \in D_\pi(x)$; we just outline this part.
As $X_t \geq 0$, we only need to show that $\int_0^t (\pi(s)X_s)^2 ds < \infty$.

$$\|\pi(t)X_t\|^2 \leq \underbrace{\left\|(\sigma(t)^{-1})^t \frac{\kappa_t}{X_t\delta(t)}\right\|^2}_{:=\nu_1} + \underbrace{\left\|(\sigma(t)^{-1})^t\xi(t)\right\|^2}_{:=\nu_2}$$

For $\nu_1 < \infty$ notice that $(\sigma^{-1})^t\sigma^{-1}$ is uniformly positive definite,
$\int_0^t \|\kappa_s\|^2 ds < \infty$ and $\delta(t) \geq 0$, $\delta(t) \in \mathcal{C}^0$.
Moreover, $\nu_2 < \infty$ because of the properties of σ^{-1}, r, b and $X_t \in \mathcal{C}^0$

$$\Rightarrow \int_0^t \|\pi(s)X_s\|^2\, ds$$
$$\Rightarrow \int_0^t (\pi(s)X_s)^2\, ds$$
$$\Rightarrow (\pi, C) \in D_\pi(x)$$

■

Proof of theorem 3.10, chapter 3.3

The proof is closely related to the one of Korn (compare Korn [2], pp. 68-70).
i) Case $x \geq \hat{x}$:

We know that
$\max U_1^t(\cdot) = x_1$ with $x_1 = I_1^t(0)$ and
$\max U_2(\cdot) = x_2$ with $x_2 = I_2(0)$.

$$\begin{aligned} \Rightarrow\quad & E\left[\int_0^T U_1^s(x_1)ds + U_2(x_2)\right] \\ =\quad & E\left[\int_0^T U_1^s(I_1^s(0))ds + U_2(I_2(0))\right] \quad P\text{-a.s.} \\ \geq\quad & E\left[\int_0^T U_1^s(C_s)ds + U_2(X^{y,\pi,C}(T))\right] \quad \text{for every } (\pi, C) \in D_\pi^*(z) \text{ with } y \leq x \end{aligned}$$

Hence, $Q^* = x_2$ and $C_t^* = x_1$ yield the optimal utility for almost all $\omega \in \Omega$ (i.e. is path-wise optimal).

Please note that the existence of $\pi^*(t)$ with $(\pi^*(t), C_t^*) \in D_\pi^*(\hat{x})$ is deduced from the completeness of the market (theorem 2.7).

As Q^* and C_t^* are deterministic we have

$$E\left[\int_0^T U_1^s(C_s^*)^- ds + U_2(X^{\hat{x},\pi^*,C^*}(T))^-\right] = E\left[\int_0^T U_1^s(x_1)^- ds + U_2(x_2)^-\right] < \infty$$

Therefore, (3.10) in section 3.3 is held and $(\pi^*(t), C_t^*) \in D_\pi^*(\hat{x})$.

Consequently, (Q^*, C_t^*) are optimal for (P) in this case.

ii) Case $x < \hat{x}$:

a) $E\left[\int_0^T \delta(s)C_s^* ds + \delta(T)Q^*\right] = x$ because of the definition of (C_t^*, Q^*).

b) As $A^{-1}(x)\delta(t) > 0 \; \forall \; t \in I$ and I_1^t, $I_2 > 0$ it follows that $C_t^* > 0$ and $Q^* > 0$.

c) The existence of a portfolio process $\pi^*(t)$ with $(\pi^*(t), C_t^*) \in D_\pi(x)$ results from the existence of a solution (C_t^*, Q^*) and theorem 2.7 (the complete market).

d) To show that $(\pi^*(t), C_t^*) \in D_\pi^*(x)$ use theorem A.10 7):

$$\begin{aligned} & U_t(I_t(y)) \geq U_t(x) + y(I_t(y) - x) \qquad (= \text{theorem A.10 7)}) \\ \hookrightarrow \quad & U_1^t(C_t^*) \geq U_1^t(1) + A^{-1}(x)\delta(t)(C_t^* - 1) \\ \hookrightarrow \quad & U_2(Q^*) \geq U_2(1) + A^{-1}(x)\delta(T)(Q^* - 1) \end{aligned}$$

With b) and $\quad a \geq b \quad \Rightarrow \quad a^- \leq b^- \leq |b| \quad$ we get

$$\begin{aligned} & E\left[\int_0^T U_1^s(C_s^*)^- ds + U_2(Q^*)^-\right] \\ \leq \; & E\left[\int_0^T \left(|U_1^s(1)| + A^{-1}(x)\delta(s)(C_s^* - 1)\right)\right] \\ & + E\left[|U_2(1)| + A^{-1}(x)\delta(T)(Q^* - 1)\right] \\ = \; & \underbrace{\int_0^T |U_1^s(1)|\, ds}_{<\infty} + \underbrace{|U_2(1)|}_{<\infty} + \\ & \underbrace{A^{-1}(x)}_{<\infty}\left(\underbrace{E\left[\int_0^T \delta(s)C_s^* ds + \delta(T)Q^*\right]}_{=x<\infty} + \underbrace{E\left[\int_0^T \delta(s) ds\right]}_{<\infty} + \underbrace{E[\delta(T)]}_{<\infty}\right) \\ & <\infty \end{aligned}$$

e) To proof that (Q^*, C_t^*) is optimal for (P) we remember the heuristic solution in chapter 3.2:

$$\begin{aligned}
& \Psi(x, \pi^*, C_t^*) \\
= \; & E\left[\int_0^T U_1^s(C_s^*)ds + U_2(Q^*)\right] \\
\geq \; & E\Big[\int_0^T U_1^s(C_s) + A^{-1}(x)\delta(s)(C_s^* - C_s)ds + U_2(X^{y,\pi,C_t}(T)) \\
& + A^{-1}(x)\delta(T)(Q^* - X^{y,\pi,C_t}(T))\Big] \\
= \; & E\left[\int_0^T U_1^s(C_s)ds + U_2(X^{y,\pi,C_t}(T))\right] \\
& + A^{-1}(x)\underbrace{E\left[\int_0^T \delta(s)C_s^* ds + \delta(T)Q^*\right]}_{=x} \\
& - A^{-1}(x)\underbrace{E\left[\int_0^T \delta(s)C_s ds + \delta(T)X^{y,\pi,C_t}(T)\right]}_{:=y\leq x} \\
\geq \; & \Psi(x, \pi, C_t) + \underbrace{A^{-1}(x)}_{\geq 0}\underbrace{(x-y)}_{\geq 0} \\
\geq \; & \Psi(x, \pi, C_t)
\end{aligned}$$

■

Proof of theorem 3.14, chapter 3.4

Analogous to Korn [2], pp. 76-77.

The proof of theorem 2.7 ii) shows that

$$\frac{1}{\delta(t)}E\left[\int_t^T \delta(s)C_s^* ds + \delta(T)Q^* | \mathcal{F}_t\right] = X(t)$$

in which (Q^*, C_t^*) are derived from theorem 3.10, is valid.

$X(t)$ also satisfies the equation (see equation (2.11) in definition 2.4 a))

$$\begin{aligned}
X(t) = x^* + \int_0^t \left[\vartheta_0(s)P_0(s)r(s) + \sum_{i=1}^n \vartheta_i(s)P_i(s)b_i(s)\right] ds \\
+ \sum_{i=1}^n \int_0^t \sum_{j=1}^n \vartheta_i(s)P_i(s)\sigma_{ij}(s)dB_j(s) - \int_0^t C_s^* ds
\end{aligned} \qquad (A)$$

On the other hand we apply the Itô formula A.8 on

$$f(t, B_1(t), \dots, B_n(t)) = \frac{1}{\delta(t)} E\left[\int_t^T \delta(s) C_s^* ds + \delta(T) Q^* | \mathcal{F}_t\right]$$

$$\begin{aligned} X(t) =& \frac{1}{\delta(t)} E\left[\int_t^T \delta(s) C_s^* ds + \delta(T) Q^* | \mathcal{F}_t\right] \\ =& f(0, \dots, 0) + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_1(s), \dots, B_n(s)) \right. \\ &+ \left. \frac{1}{2} \sum_{i,j=1}^n \frac{\partial f}{\partial x_i \partial x_j}(s, B_1(s), \dots, B_n(s))\right) ds \\ &+ \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(s, B_1(s), \dots, B_n(s)) dB_i(s) \end{aligned} \qquad (B)$$

As (A) and (B) are both representations for the conditional expectation above, they must be equal.

With $f(0, \dots, 0) := x^*$ and $\varphi(t) := (\vartheta_1(t)P_1(t), \dots, \vartheta_n(t)P_n(t))^t$ we compare the integrands of the stochastic integrals in (A) and (B) and receive:

$$\begin{aligned} \sum_{i,j=1}^n \vartheta_i(t) P_i(t) \sigma_{ij}(t) =& \quad \varphi(t)^t \sigma(t) \\ =& \quad \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, B_1(t), \dots, B_n(t)) \\ =& \quad \nabla_B f(t, B_1(t), \dots, B_n(t)) \end{aligned}$$

As $\sigma(t)$ is invertible, we have

$$\begin{aligned} & \varphi(t) = \left(\sigma(t)^{-1}\right)^t \cdot \nabla_B f(t, B_1(t), \dots, B_n(t))^t \\ \Rightarrow \quad & \varphi_i(t) = \vartheta_i(t) P_i(t) = \left(\left(\sigma(t)^{-1}\right)^t \cdot \nabla_B f(t, B_1(t), \dots, B_n(t))^t\right)_i \\ \Rightarrow \quad & \vartheta_i^*(t) = \frac{1}{P_i(t)} \left(\left(\sigma(t)^{-1}\right)^t \cdot \nabla_B f(t, B_1(t), \dots, B_n(t))^t\right)_i \qquad \text{for } i = 1, \dots, n \end{aligned}$$

Furthermore, we know that

$$\vartheta_0^*(t) = \frac{X(t) - \sum_{i=1}^n \vartheta_i^*(t) P_i(t)}{P_0(t)}$$

is always satisfied.

Analogous to definition 2.3 b) we receive a representation for π^* (respectively for $\pi_i^*(t)$ and $i = 1, \ldots, n$):

$$\pi_i^*(t) = \frac{\vartheta_i(t) P_i(t)}{X(t)} = \frac{\left(\left(\sigma(t)^{-1}\right)^t \cdot \nabla_B f(t, B_1(t), \ldots, B_n(t))^t \right)_i}{X(t)}$$

$$\Rightarrow \quad \pi^*(t) = \frac{1}{X(t)} \left(\sigma(t)^{-1}\right)^t \cdot \nabla_B f(t, B_1(t), \ldots, B_n(t))^t$$

In addition to that $\vartheta^*(t)$ fulfills the assumptions of definition 2.4 by definition. Thus, we have found the optimal trading strategy ϑ^* with the accompanying portfolio process π^*. ■

Bibliography

[1] H. Markowitz: *Portfolio Selection*, Journal Of Finance, 1952, pp. 77-91

[2] R. Korn: *Optimal Portfolios*, World Scientific, 1997

[3] M. Günther und A. Jüngel: *Finanzderivate mit MATLAB*, Vieweg, 2003

[4] R. Korn und E. Korn: *Optionsbewertung und Portfoliooptimierung*, Vieweg, 2001

[5] B. Øksendal: *Stochastic Differential Equations*, Springer, 2003

[6] H. Bauer: *Wahrscheinlichkeitstheorie*, de Gruyter, 2002

[7] H. Bauer: *Maßtheorie*, de Gruyter, 1992

[8] H. Heuser: *Lehrbuch der Analysis Teil II*, Teubner, 2000

[9] R. Michel: *Maßtheorie*, lecture WS 2003/2004

[10] R. Michel: *Wahrscheinlichkeitstheorie*, lecture SS 2004

[11] I.Karatzas, J.P. Lehoczky, S.E. Shreve: *Optimal portfolio and consumption decisions for a 'small investor' on a finite horizon*, SIAM Journal On Control And Optimization 27, 1987, pp. 1157-1186

[12] R.C. Merton: *Lifetime portfolio selection under uncertainity: the continuous case*, Reviews Of Economical Statistics 51, 1969, pp. 247-257

[13] R.C. Merton: *Optimum consumption and portfolio rules in a time continuous time model*, Journal Of Economic Theory 3, 1971, pp. 373-413

[14] D. Occone, I. Karatzas: *A generalized Clark representation formula with application to optimal portfolios*, Stochastics And Stochastics Reports 34, 1991, pp. 187-228

[15] P. Billingsley: *Convergence of probability measures*, J. Wiley & Sons, 1969

[16] I. Karatzas, S.E. Shreve: *Brownian motion and stochastic calculus*, Springer, 1991