Prof. Dr. Michael Günther
Zuzana Bučková, M.Sc.
Dmitry Shcherbakov, M.Sc.
Dipl. Math. Michèle Wandelt, M.Sc.
Winter Term 2014/15

Bergische Universität Wuppertal
Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Numerical Analysis and Simulation I: Ordinary Differential Equations (ODEs)

Sheet 11 - Numerical Integration of SDEs

Return of the Exercise Sheet: Tuesday, January 20th before the lecture

Examination: Tuesday, February 10, 10:00-12:00 in Hörsaal 4 (F.10.01)

Exercise 23: *Euler-Maruyama Scheme*
We investigate the Euler-Maruyama scheme. Consider the stochastic differential equation

$$dx_t = a(x_t, t)dt + b(x_t, t)dW_t \ . \tag{1}$$

a) Set up the Euler-Maruyama scheme for the SDE (1).

b) Assume that initial values are given. How can the Wiener process be realized?

c) Sketch a standard normal distribution with mean 0 and variance 1.

d) Sketch a Wiener process.

Exercise 24: *Exact Solution of a Stochastic Differential Equation*
Consider the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t \ . \tag{2}$$

Prove that the exact solution of (2) is given by the Brownian motion

$$X_t = X_0 \exp\Big(\big(\mu - \frac{1}{2}\sigma^2\big)t + \sigma W_t\Big) \ .$$

Hint: Apply Itôs lemma at

$$X_t = f(Y_t, t) = \big(\mu - \frac{1}{2}\sigma^2\big)t + \sigma Y_t$$

with $Y_t = W_t$, $a = 0$ and $b = 1$.

Exercise 25: *Strong and Weak Convergence*
We investigate the strong and weak convergence of a stochastic differential equation

$$dx_t = a(x_t, t)dt + b(x_t, t)dW_t \ .$$

a) How are the strong and weak convergence defined?

b) What is the difference between the strong and weak convergence?

c) Assumed that you want to check the strong and weak convergence of a numerical approximation, how do you proceed? Please sketch an algorithm for both cases.

Homework 21: *Euler-Maruyama Scheme* (10 Points)
We investigate the Euler-Maruyama scheme numerically. For this purpose, consider the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t \qquad (*)$$

and write a program (for example, in MATLAB) that plots 5 numerical approximations of X_t and its 5 different associated Wiener processes. Proceed as follows:

a) Set up the Euler-Maruyama scheme for the SDE (*). How can the Wiener process be realized?

b) Write a program that computes one solution of the SDE (*). The Wiener process W can be generated by the command

*h=0.01; n=10/h; dW = randn(1,n) * sqrt(h); W = [0,cumsum(dW)];*

c) Plot 5 different Wiener processes and its appropriate solutions for the SDE (*).
Use $\mu = 2, \sigma = 1, X_0 = 1$, stepsize $h = 0.01$ and compute the SDE up to $t_{end} = 10$.

Please hand in your results with plots and source code (you can also send it by email).

Homework 22: *Euler-Maruyama Scheme* (10 Points)
We investigate the Euler-Maruyama scheme numerically concerning strong and weak convergence. For this purpose, consider the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t \qquad (*)$$

and write a program (for example, in MATLAB) that plots the strong and weak convergence for different step sizes. Proceed as follows:

a) Write a program that computes M different Wiener processes and its appropriate solutions for the SDE (*). (You can re-use your program of homework 21.)

b) Show that the Euler-Maruyama scheme has the weak convergence order $\gamma = 1$:
Compute M solutions of the SDE (*) using different step sizes h. At the end, produce a double-logarithmic plot (with the command loglog) of the step size and $\epsilon(h)$.

Hint: The weak convergence order β is computed via

$$\epsilon(h) \approx c \cdot h^{\beta} \qquad \text{with} \qquad \epsilon(h) := |E(X_T^h) - E(X_n^h)|$$

being the absolute value of the difference of the expectation values of the exact solutions X_T^h and the numerical approximations X_n^h for different step sizes. The exact solution is given in exercise 24.

c) Compute the strong convergence order γ in simulating M solutions of the SDE (*) using different step sizes h. Produce a double-logarithmic plot (with the command loglog) of the step size and $\epsilon(h)$. Ensure that $\gamma = \frac{1}{2}$ holds.

Hint:The strong convergence order γ is computed via

$$\epsilon(h) \approx c \cdot h^{\gamma} \qquad \text{with} \qquad \epsilon(h) := E(|X_T - X_n|)$$

being the expectation value of the difference of the exact solution X_T^j and the numerical approximation X_n^j for all processes $j = 1, \ldots, M$. So, you need again the exact solution of the SDE (*) here.

Some hints for the parameters:

- you can fix the variables as $\mu = 2, \sigma = 1, X_0 = 1$,
- a reasonable value for M is 1000,
- the step size can be set as $h = 2^{-n}, n = 1, \ldots, 6$,
- and t_{end} as $t_{end} = 1$.

Please hand in your results with plots and source code (you can also send it by email).