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Numerical Analysis and Simulation I: Ordinary Differential Equations (ODEs)

Sheet 8 - Differential Algebraic Equations

Return of the Exercise Sheet: Tuesday, December 16th before the lecture

Exercise 17: *Multistep methods for general DAEs*
A linear multistep method for an ODE-IVP $y'(x) = f(x, y(x))$, $y(x_0) = y_0$ exhibits the formula

$$\sum_{l=0}^{k} \alpha_l y_{i+l} = h \sum_{l=0}^{k} \beta_l f(x_{i+l}, y_{i+l})$$

with coefficients $\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_k$ and equidistant step size h. The solutions $y_i, y_{i+1}, \ldots, y_{i+k-1}$ have already been computed, such that y_{i+k} is the only unknown.

We introduce $z_{i+l} := f(x_{i+l}, y_{i+l})$ such that the linear multistep scheme can be rewritten as

$$\sum_{l=0}^{k} \alpha_l y_{i+l} = h \sum_{l=0}^{k} \beta_l z_{i+l}$$

with only unknowns y_{i+k} and z_{i+k}. Then, the general nonlinear system of differential algebraic equations is described by

$$F(z_{i+k}, y_{i+k}, x_{i+k}) = 0 \ .$$

The BDF methods

$$\sum_{l=0}^{k} \alpha_l y_{i+l} = h z_{i+k}$$

are suitable for solving systems of DAEs. Please apply the methods BDF-2 and BDF-3.

Hint: Replace the unknown z_{i+k} in $F(z_{i+k}, y_{i+k}, x_{i+k}) = 0$ by the formulas obtained via BDF-2 and BDF-3.

Exercise 18: *Multistep methods for semi-explicit DAEs*
For an autonomous system of ODEs $y' = f(y)$, a linear multistep method exhibits the formula

$$\sum_{l=0}^{k} \alpha_l y_{i+l} = h \sum_{l=0}^{k} \beta_l f(y_{i+l})$$

with coefficients $\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_k$ and equidistant step size h.
Use the direct approach (ε-embedding) and the indirect approach (state space form) to obtain two types of methods for semi-explicit systems of DAEs with index 1

$$\begin{aligned} y'(x) &= f(y(x), z(x)), \\ 0 &= g(y(x), z(x)). \end{aligned}$$

Compare the two techniques. Which approach should be preferred?

Homework 15: *ROW methods for semi-explicit DAEs* (10 Points)
For an autonomous system of ODEs $y' = f(y)$, a Rosenbrock-Wanner method reads

$$k_i = f\Big(y_0 + h\sum_{j=1}^{i-1}\alpha_{ij}k_j\Big) + hJ\sum_{j=1}^{i}\gamma_{ij}k_j, \qquad i = 1, \dots, s$$
$$y_1 = y_0 + h\sum_{i=1}^{s} b_i k_i$$

with coefficients $b_i, \alpha_{ij}, \gamma_{ij}$ ($\gamma_{ii} = \gamma > 0$ for all i) and the Jacobian matrix $J = \partial f/\partial y$.
Apply the direct approach (ε-embedding) to obtain the formula of the Rosenbrock-Wanner method in case of semi-explicit DAEs

$$y'(x) = f\big(y(x), z(x)\big),$$
$$0 = g\big(y(x), z(x)\big).$$

Thereby, specify the formulas for the determination of the increments k_i and l_i with respect to y and z, respectively. Verify that the resulting linear system is uniquely solvable for sufficiently small step size h in the index-1 case.

Homework 16: *One step Method* (2+2+1+2+3 Points)
Consider the initial value problem (IVP)

$$y' = y^2, \quad x > 0, \qquad y(0) = 1. \tag{IVP}$$

a) What can you state about the (local/global) existence and uniqueness of solutions to (IVP)? (roughly 3 sentences)

b) The *improved Euler scheme* reads

$$y_1 = y_0 + \frac{h}{2}\big[f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))\big]. \tag{iEUL}$$

Sketch the used integration idea and explain which approximations are used.

c) What is the *increment function* $\Phi(x, y, h; f)$ here? State it explicitly for the given IVP above.

d) Calculate the first step of the improved Euler scheme (iEUL) with an arbitrary step size $h > 0$. State the formula for the second step for this special case!

e) Determine the stability function $R : \mathbb{C} \to \mathbb{C}$ for the improved Euler scheme considering Dahlquist's test equation. Which part of the real axis belongs to the stability domain. Is this scheme A-stable or L-stable? Why?