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Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Numerical Analysis and Simulation I: Ordinary Differential Equations (ODEs)

Sheet 5 - Multistep Methods

Return of the Exercise Sheet: Tuesday, November 25th before the lecture

Exercise 10: *Consistency and Stability of the Adams-Bashforth Method*
Consider the *Adams-Bashforth* method

$$y_{i+1} = y_i + h \sum_{j=1}^{2} \gamma_j f(x_{i-j+1}, y_{i-j+1}). \tag{ABM}$$

The local discretisation error $\tau(h)$ of this method can be written as

$$\tau(h) = \frac{1}{h} \left(y(x_{i+1}) - y(x_i) - h \sum_{j=1}^{2} \gamma_j y'(x_{i-j+1}) \right).$$

a) Derive conditions for the coefficients γ_j, $j = 1, 2$ to achieve consistency of order 2.

b) Investigate the numerical stability of the scheme (ABM). Make a statement about the convergence of the method?

Exercise 11: *Analysis of Multistep Methods*
Investigate the convergence of the following multistep methods by analysing the consistency and the stability. Determine the order of convergence if the answer is positive.

a) $y_{j+1} - y_{j-1} = 2hf(x_j, y_j)$ (explicit midpoint rule)

b) $y_{j+1} - y_{j-1} = hf(x_{j+1}, y_{j+1})$

c) $\frac{3}{2}y_{j+1} - 2y_j + \frac{1}{2}y_{j-1} = hf(x_{j+1}, y_{j+1})$ (BDF-2)

d) $\frac{1}{2}y_{j+1} - 2y_j + \frac{3}{2}y_{j-1} = -hf(x_{j-1}, y_{j-1})$

Homework 9: *Adams-Moulton – Trapezoidal Rule* (10 Points)
In order to derive multi-step schemes of Adams type, we consider the differential equation $y' = f(x, y)$ which we integrate from x_i up to x_{i+1},

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(x, y(x)) \, \mathrm{d}x.$$

The integrand is replaced by a polynomial $p(x)$, which interpolates f_{i-k+j} for $j = 1, \ldots, k+1$.
In this exercise, we want to derive the trapezoidal rule

$$y_{i+1} = y_i + h \left(\frac{1}{2} f_i + \frac{1}{2} f_{i+1} \right)$$

explicitly, which is obtained for the special case $k = 1$. For this reason, we construct the interpolation polynomial $p(x)$ for the given equidistant grid $x_i = x_0 + i \cdot h$. Use Lagrangian polynomials for computing the interpolation polynomial and solve the resulting integrals in detail.

Homework 10: *Multistep Method* (10 Points)
Let the method

$$y_{i+2} = 4y_{i+1} - 3y_i - 2hf(x_i, y_i)$$

be given.

a) Analyse the method for consistency.

b) Analyse the method for stability.

c) Consider the problem

$$y' = 0, \quad x \in (a, b) \text{ with arbitrary initial value } y(a) = c.$$

Let the starting values $y_0 = c$ and $y_1 = c + \epsilon$ be given, where ϵ denotes a small error. Perform three steps of the numerical scheme given above. Compare your results to the analytical solution. Is the observation in line with your findings in parts **a)** and **b)**?