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Fachbereich C – Mathematik und Naturwissenschaften
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Numerical Analysis and Simulation I: Ordinary Differential Equations (ODEs)

Sheet 2 - Theory of ODEs + Elementary Integration Schemes

Return of the Exercise Sheet: Tuesday, November 4th before the lecture

Exercise 3: *Initial and boundary value problem*
For the linear two-point boundary value problem

$$y''(x) + y(x) = 0, \qquad (y : \mathbb{R} \to \mathbb{R})$$

we define the following different boundary conditions

(i) $y(0) = y(\pi/2) = 1$,

(ii) $y(0) = y(\pi) = 1$,

(iii) $y(0) = y(2\pi) = 1$.

a) Solve the differential equation with the initial values $y(0) = a,\ y'(0) = b$.

b) Discuss the solution for the different boundary value problems.

c) Compare the existence and uniqueness for initial and boundary value problems.

Exercise 4: *Picard-Lindelöf-Iteration*
The proof of the Picard-Lindelöf theorem (existence and uniqueness of the initial value problem $y' = f(x, y)$ with $y(x_0) = y_0$) uses a fixed point iteration

$$y_{i+1} = y_0 + \int_{x_0}^{x} f(t, y_i(t))\, dt$$

with $i \in \mathbb{N}$ and initial value $y_0(x) = y_0$. Then, the convergence of the iteration follows from the fixed point theorem of Banach.

a) Show that the requirements of the Picard-Lindelöf theorem are satisfied for the IVP

$$y' = 2xy, \qquad x \in [a, b].$$

b) Calculate the first two steps $y_1(x)$ and $y_2(x)$ of the Picard-Lindelöf iteration with initial value $y(0) = c$. *Hint: The initial condition $y(0) = c$ implies $x_0 = 0$.*

c) Verify that

$$y_k(x) = c\left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \ldots + \frac{x^{2k}}{k!}\right).$$

Hint: Use mathematical induction.

d) Calculate the limit

$$y(x) = \lim_{k\to\infty} y_k(x).$$

To which function does $y(x)$ converge?

Homework 3: *Consistency of one-step schemes* (10 Points)
We discuss explicit one-step schemes to solve initial value problems (IVPs)

$$y' = f(x, y), \quad y(x_0) = y_0 \qquad (y : \mathbb{R} \to \mathbb{R}).$$

Explicit one-step schemes are defined by

$$y_{i+1} = \Phi(x_i, y_i, h), \qquad i = 0, 1, \ldots$$

with the increment function Φ, discrete values x_i, y_i and the (in this case constant) step size $h = x_{i+1} - x_i$.

We consider the following one-step schemes:

(i) explicit Euler scheme $\quad u_1 = u_0 + hf(x_0, u_0)$

(ii) explicit Euler scheme with half step size

$$v_{1/2} = v_0 + \tfrac{h}{2} \cdot f(x_0, v_0), \qquad v_1 = v_{1/2} + \tfrac{h}{2} \cdot f\big(x_0 + \tfrac{h}{2}, v_{1/2}\big)$$

(iii) modified Euler scheme according to Collatz

$$w_1 = w_0 + hf\big(x_0 + \tfrac{h}{2}, w_0 + \tfrac{h}{2} \cdot f(x_0, w_0)\big).$$

a) Determine the order of consistency for the schemes (i) and (ii). How are the leading terms in $\mathcal{O}(h^{p+1})$ of the two schemes related?

b) Show $w_1 = 2v_1 - u_1$ for $u_0 = v_0 = w_0 = y_0$. Which conclusion can be drawn for the order of consistency of the modified Euler scheme (iii)?

c) Illustrate the three schemes geometrically.

Remind: The two-dimensional Taylor expansion (for f sufficiently smooth) reads

$$f(x + h_x, y + h_y) = f(x, y) + h_x \tfrac{\partial f}{\partial x}(x, y) + h_y \tfrac{\partial f}{\partial y}(x, y) + \mathcal{O}(h_x^2) + \mathcal{O}(h_x h_y) + \mathcal{O}(h_y^2).$$

Homework 4: *Gronwall's Lemma* (10 Points)
Prove Gronwall's Lemma:
Assume that $m(x)$ is a positive, continuous function and that $\rho \geq 0, \epsilon \geq 0$.
Then the integral inequality

$$m(x) \leq \rho + \epsilon(x - x_0) + L \int_{x_0}^{x} m(t)dt,$$

implies the estimate

$$m(x) \leq \rho e^{L(x-x_0)} + \frac{\epsilon}{L}(e^{L(x-x_0)} - 1).$$

Hint: Introduce the function $q(x) = e^{-Lx} \int_{x_0}^{x} m(t)dt$. Then, proceed the following steps: compute q', solve for $m(x)$ and get equation (1). Then, insert (1) in the integral inequality to obtain the expression $e^{Lx} q'(x) \leq \rho + \epsilon(x - x_0)$. From this equation, you can get $q'(x)$ and via integration $q(x)$. If you put these results in (1), you are done. Please do not copy the proof from the lecture notes without further explanations.