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STRUCTURE PRESERVING STOCHASTIC INTEGRATION SCHEMES IN INTEREST RATE DERIVATIVE MODELING

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Abstract: In many applications, differential equation models require geometric integration, i.e., the application of structure-preserving integration schemes. In computational finance, for example, the numerical simulation of extended Libor market models used to value structured interest rate derivatives has to preserve positivity or boundedness of the underlying stochastic processes used to model mean-reverting volatility or forward rates. This paper discusses how stochastic integration schemes can be constructed in order to maintain these properties of the analytical solution. Milstein-type methods prove to be the method-of-choice with respect to both efficiency and preservation of structural properties, as they turn out to dominate the increments of Brownian motions. These theoretical results are confirmed by numerical tests.

Key words: stochastic differential equation, structure-preserving, positivity, eternal life span, Milstein schemes, Libor market models, mean reverting, forward rate model, displaced diffusion, constant elasticity of variance.

1 Introduction

Geometric integration, i.e., development and analysis of structure-preserving integration schemes, is indispensable in the numerical simulation of many ordinary differential and differential-algebraic models [4]. In many applications, essential model properties such as (equality and inequality) constraints or positivity are mandatory and have to be maintained in the numerical approximation. Positivity in particular plays an essential role for some processes in financial applications. For example stock option models or state-of-the-art interest rate derivative models usually require positivity due to financial interpretation.

In the extended Libor market model with stochastic volatility [1, 2, 7], forward rates F_t are defined by scalar stochastic differential equations of the type

$$dF_t = \sigma(t)\sqrt{V_t}\varphi(F_t)dW_t, \qquad F_0 > 0 \tag{1}$$

where W_t denotes a Brownian motion on a probability space $(\Omega, \mathbb{F}, P)$ and $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ a positive function. The function φ can be given either by displaced diffusion, $\varphi(x) = x + m$ ($m \in \mathbb{R}$), or as constant elasticity of variance, $\varphi(x) = x^\alpha$ ($\alpha \in \mathbb{R}^+$). The volatility V_t is described by the mean reverting process

$$dV_t = \kappa(\theta - V_t)dt + \epsilon V_t^p d\widetilde{W}_t, \qquad V_0 > 0 \tag{2}$$

with $\kappa, \theta, \epsilon \in \mathbb{R}^+$, $1/2 < p \leq 1$ and a second Brownian motion $\widetilde{W}_t$ uncorrelated to W_t. The probability space satisfies the usual conditions and the filtration is assumed to be generated by the two Wiener processes $\mathcal{F} = \sigma(W_t, \widetilde{W}_t)$. This extended model (1–2) does not allow for an analytic solution and forward rates are non-lognormal, introducing a smile into the implied volatility surface. As the mean reverting process (2) for V_t can be computed independently of (1), we have to deal with two (only one-sided) coupled scalar stochastic differential equations.

Depending on the model parameters, one can show that the volatility V_t and the forward rates F_t are bounded from below by $\bar{b} \in \mathbb{R}$, i.e.,

- $V_0 > 0 \Longrightarrow P\left(\left\{V_t > \bar{b} \text{ for all } t > 0\right\}\right) = 1$ with $\bar{b} = 0$;
- $F_0 > 0 \Longrightarrow P\left(\left\{F_t \geq \bar{b} \text{ for all } t > 0\right\}\right) = 1$ with $\bar{b} = -m$ (displaced diffusion) and $\bar{b} = 0$ (constant elasticity of variance), resp.;

These structural properties are generally required in order to interpret V_t as a volatility and F_t as a forward rate and have to be preserved during numerical integration. Based on an analysis of analytical and numerical positivity, we show that this task can be performed efficiently by stochastic integration schemes of the Milstein type.

The paper is organized as follows. Introducing the concept of analytical positivity and boundedness from below (by a constant $\bar{b} \in \mathbb{R}$), both model equations (1–2) are classified in Section 2 with respect to structural properties depending on model parameters. Following the lines of Schurz [11], the notion of eternal (or finite) life time of stochastic integration schemes is used to test these schemes for numerical positivity and boundedness, resp., if applied to (1) and (2). Whereas standard approaches such as the Euler method must fail, schemes that are based on balancing or dominating the Wiener increment $\Delta W_{t_{n+1}} := W_{t_{n+1}} - W_{t_n}$ allow for structure-preservation. The analysis indicates that the Milstein method, a dominating scheme, turns out to be the best in any case: its explicit version for forward rate model (1), and its implicit one for mean-reverting processes (2). Balancing schemes, however, are characterized by a conflict of interest: if not restricting to a weaker version of numerical positivity, numerical positivity excludes convergence of the method and vice versa. Numerical tests in Section 4 for both model equations validate these theoretical results.

2 Analytical positivity and boundedness

Consider a scalar Itô diffusion X_t given by the stochastic differential equation

$$dX_t = a(X_t)dt + b(X_t)dW_t, \quad X_0 > \bar{b} \tag{3}$$

with Brownian motion W_t. To decide, whether X_t is bounded from below by $\bar{b}$ or not, we utilize the concepts introduced in [9] without giving any proofs, which would clearly extend the scope of this article.

We define the scale function s and the speed density m,

$$s(x) = \exp\left(-\int_{x_0}^{x} \frac{2a(t)}{b^2(t)}dt\right), \qquad m(x) = \frac{1}{s(x)b^2(x)}$$

and correspondingly the measures $dM = m(x)dx$ and $dS = s(x)dx$. In terms of these differentials the generator L of the Itô diffusion (3) is given by

$$Lf(x) = \frac{1}{2}\frac{d}{dM}\left[\frac{df}{dS}(x)\right]. \tag{4}$$

Based on the knowledge of drift a and diffusion b, this representation enables us to classify the range of an Itô diffusion X_t with respect to a boundary $\bar{b} \in \mathbb{R}$:

Definition 2.1 *Let $x > \bar{b}$ be arbitrary but fixed, and*

$$S(\bar{b}, x] := \lim_{z \to \bar{b}} S[z, x], \qquad S[z, x] := \int_z^x s(y)dy, \tag{5}$$

$$\Sigma(\bar{b}) := \int_{\bar{b}}^{x} S(\bar{b}, z]dM(z) = \int_{\bar{b}}^{x} M[z, x]dS(z). \tag{6}$$

Then the boundary $\bar{b}$ is called attractive if

$$S(\bar{b}, x] < \infty.$$

We classify $\bar{b}$ as attainable if

$$\Sigma(\bar{b}) < \infty.$$

Assuming $X_0 > \bar{b}$, these concepts can be interpreted as follows:

- The Itô diffusion X_t has an unattractive or attractive but unattainable bound $\bar{b}$, if
$$P\left(\left\{X_t > \bar{b} \text{ for all } t > 0\right\}\right) = 1.$$
- The Itô diffusion X_t has an attainable bound $\bar{b}$, if
$$P\left(\left\{X_t \geq \bar{b} \text{ for all } t > 0\right\}\right) = 1 \text{ and } P\left(\left\{\exists t^\star > 0 : X_{t^\star} = \bar{b}\right\}\right) > 0;$$

For $\bar{b} = 0$, attractivity and attainability are equivalent to positivity and non-negativity.

This characterization enables us to show the assumed structural characteristics of forward rates (1) and for the volatility (2) asserted in the introduction.

Mean reverting processes

Depending on its parameters, the mean reverting process (2) has a positive (non-negative range):

Lemma 2.2 *The stochastic process given by*

$$dX_t = \kappa(\theta - X_t)dt + \epsilon X_t^p dW_t \tag{7}$$

with $\kappa, \theta, \epsilon, p \in \mathbb{R}^+$*, has an unattractive bound* $\bar{b} = 0$ *if*

1. $p = \frac{1}{2}$ *and* $2\kappa\theta > \epsilon^2$ *or*
2. $p > \frac{1}{2}$

otherwise zero is attainable.

Proof: To verify the first assertion, we start with computing the function s

$$\begin{aligned} s(x) &= \exp\left(-\int_{x_0}^{x} \frac{2\kappa(\theta - z)}{\epsilon^2 z} dz\right) = \exp\left(-\log(x)\frac{2\kappa\theta}{\epsilon^2} + x\frac{2\kappa}{\epsilon^2}\right) \\ &= x^{-2\kappa\theta/\epsilon^2} \exp\left(x\frac{2\kappa}{\epsilon^2}\right). \end{aligned}$$

This calculation leads to the following result for the measure S

$$S(0, x] = \infty \text{ if } 2\kappa\theta > \epsilon^2 \text{ and } S(0, x] < \infty \text{ else.}$$

Accordingly we only have to check the case where $2\kappa\theta \leq \epsilon^2$ (to simplify notation let $\lambda = 2\kappa\theta/\epsilon^2$):

$$\Sigma(0) = \int_0^x \left(\int_0^y s(z)dz\right) m(y)dy = \int_0^x C y^{1-\lambda} \frac{y^\lambda}{\epsilon^2 y} dy = \int_0^x C/\epsilon^2 dy < \infty.$$

Thus the lower bound 0 is always attainable if $2\kappa\theta \leq \epsilon^2$.

To prove the second statement, we have to calculate the function s:

$$s(y) = \exp\left(-\frac{2\kappa\theta}{\epsilon^2(1-2p)} y^{1-2p}\right) \exp\left(\frac{2\kappa}{\epsilon^2(2-2p)} y^{2-2p}\right).$$

In the case that $p > \frac{1}{2}$ the second term is bounded from below by $c \in \mathbb{R}$ on $[0, x]$. However there exists some $x_0 > 0$ such that

$$\exp\left(-\frac{2\kappa\theta}{\epsilon^2(1-2p)} x_0^{1-2p}\right) = x_0^{-1}.$$

Hence we get for any arbitrary $x > x_0$

$$S(0, x] > S(0, x_0] \geq \int_0^{x_0} c \exp\left(-\frac{2\kappa\theta}{\epsilon^2(1-2p)} y^{1-2p}\right) dy > \int_0^{x_0} y^{-1} dy = \infty.$$

Therefore the bound is unattractive. Last we have to prove that 0 is attainable if $p < \frac{1}{2}$. First we notice that $s(y) \in [L, U] \subset \mathbb{R}$ on the interval $[0, x]$. So we obtain

$$\Sigma(0) = \int_0^x \left(\int_0^y s(z)dz \right) m(y)dy \quad \leq \quad \int_0^x (Uy)\, m(y)dy \leq \frac{Ux}{L} \int_0^x \frac{1}{b(y)^2} dy < \infty$$

which completes the proof. □

Forward rate models

Assuming a positive volatility V_t and $\sigma(t) > 0$ for all $t > 0$, the positivity of the forward rates (1) is characterized by the following two corollaries that follow immediately from Def. 2.1:

Corollary 2.3 (Displaced diffusion) *The Itô diffusion*

$$dX_t = (a + bX_t)dW_t, \qquad X_0 > 0 \tag{8}$$

with $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$ has the following characteristics:

- $X_t \in [-a/b, \infty]$
- $-a/b$ *is an attractive boundary but not attainable.*

Corollary 2.4 (Constant elasticity of variance) *The Itô diffusion*

$$dX_t = X_t^\alpha dW_t, \qquad X_0 > 0 \ , \ \alpha \geq 0. \tag{9}$$

possesses the following properties

- $0 < \alpha < 1/2 \Rightarrow 0$ *is an attainable boundary,*
- $1/2 \leq \alpha \Rightarrow 0$ *is an unattainable but attractive boundary.*

The proof can be found in [8].

3 Numerical positivity and boundedness

The concept of numerical positivity was introduced by Schurz [11]. We slightly generalize this concept to boundedness from below by $\bar{b} \in \mathbb{R}$.

Definition 3.1 *Let X_t be a stochastic process with*

$$P\left(\{X_t > \bar{b} \text{ for all } t > 0\}\right) = 1. \tag{10}$$

Then the stochastic integration scheme possesses an eternal life time if

$$P\left(\{X_{n+1} > \bar{b} | X_n > \bar{b}\}\right) = 1. \tag{11}$$

Otherwise it has a finite life time.

In this definition the life time only depends on one single step of the integration scheme. Schurz also introduced a weaker definition of positivity.

Definition 3.2 *A stochastic integration scheme is called ε-positive or, more generally, ε-bounded from below by $\bar{b} \in \mathbb{R}$, if*

$$P\left(\{X_{n+1} > \bar{b} | X_n > \bar{b} + \varepsilon\}\right) = 1. \tag{12}$$

Obviously an eternal life time implies ε-boundedness.

In general, one cannot expect numerical schemes to preserve positivity or boundedness. An intrinsic example is given by

Proposition 3.3 *The Euler method has a finite life time for all Itô diffusions (3) with positive paths.*

Proof: Consider one integration step of the Euler scheme from $X_n \approx X_{t_n}$ to $X_{n+1} \approx X_{t_{n+1}}$ with $\Delta_{t_n} := t_{n+1} - t_n$ and $\Delta W_{t_n} := W_{t_{n+1}} - W_{t_n}$:

$$X_{n+1} = a(X_n)\Delta_{t_n} + b(X_n)\Delta W_{t_n}.$$

It suffices to prove the proposition for $a(X_n)b(X_n) > 0$. Then

$$X_{n+1} < 0 \Leftrightarrow \Delta W_{t_n} < -\frac{a(X_n)}{b(X_n)}\Delta_{t_n}.$$

occurs with a positive probability. □

The Euler scheme has to fail since ΔW_{t_n} takes all values $c \in \mathbb{R}$ with a positive probability. So the question raises up how it is possible to prevent a numerical integration scheme from becoming negative (or more general, bounded below). There are mainly two possibilities to provide positivity: balancing and dominating.

The idea of balancing was also introduced by Schurz et al. [10], where the Balanced Implicit Method (BIM) is used to preserve positivity.

Definition 3.4 (BIM) *One integration step of the BIM is given as follows*

$$\begin{aligned} X_{n+1} &= X_n + a(X_n)\Delta_{t_n} + b(X_n)\Delta W_{t_n} + (X_n - X_{n+1})C_n(X_n) \\ C_n(X_n) &= c_0(X_n)\Delta_{t_n} + c_1(X_n)|\Delta W_{t_n}|. \end{aligned}$$

In this method the functions c_0 and c_1 are called control functions. To guarantee convergence of the method, the control functions must be bounded and have to satisfy the inequality

$$1 + c_0(X_n)\Delta_{t_n} + c_1(X_n)|\Delta W_{t_n}| > 0. \tag{13}$$

Next we present some result about positivity preservation in the case of mean reverting processes.

3.1 Mean-reverting processes & implicit Milstein schemes

For arbitrary parameters, it is not possible to get a closed solution form for mean-reverting processes (2). First we will concentrate on the most popular mean-reverting process which are commonly used to model stochastic volatility [5, 6].

Recently these processes have been introduced to model stochastic volatility in the context of extended Libor market models [2]:

$$dX_t = \kappa(\theta - X_t)dt + \epsilon\sqrt{X_t}dW. \tag{14}$$

In Section 2 we have already analyzed this process and proved its positivity in the case that $2\kappa\theta > \epsilon^2$. One can show that the BIM method is not an appropriate choice to integrate this SDE: numerical positivity is contradictory to convergence, since positivity would demand an unbounded control function c_1. However, we can guarantee positivity in the weaker sense of ε-positivity:

Lemma 3.5 *The BIM method is ε-positive for the stochastic process* (14) *if we choose the control functions as follows:*

$$\begin{aligned} c_0 &= \kappa, \\ c_1(x) &= \begin{cases} \epsilon x^{-\frac{1}{2}} & : \textit{ if } x > \varepsilon, \\ \epsilon \varepsilon^{-\frac{1}{2}} & : \textit{ else.} \end{cases} \end{aligned}$$

Proof: See [8] and [11] for a detailed discussion. □

But we can apply the idea of domination to this SDE, regarding the class of explicit and (drift-)implicit Milstein schemes: applied to the Itô diffusion, these schemes read

$$X_{n+1} = X_n + a(\bar{X})\Delta_{t_n} + b(X_n)\Delta W_{t_n} + \frac{1}{2}b'(X_n)b(X_n)\left((\Delta W_{t_n})^2 - \Delta_{t_n}\right), \tag{15}$$

where the drift term $a(\bar{X})$ is evaluated at the approximate solution at time point t_n and t_{n+1}, respectively:

$$\bar{X} = X_n \qquad \text{(explicit Milstein scheme) ;} \tag{16}$$

$$\bar{X} = X_{n+1} \qquad \text{(implicit Milstein scheme) ;} \tag{17}$$

Due to the negative sign of X_t in the drift $a(X_t) = \kappa(\theta - X_t)$, the explicit Milstein scheme is not the right choice for (14). A more suitable choice is the implicit Milstein method, as shown in

Theorem 3.6 *The implicit Milstein method (15,17) has an eternal life time for the integration of the stochastic process* (14) *independent of the stepsize Δ_{t_n}.*

Proof: One integration step reads as follows

$$X_{n+1} = X_n + \kappa(\theta - X_{n+1})\Delta_{t_n} + \epsilon X_n^{\frac{1}{2}} \Delta W_{t_n} + \frac{1}{4}\epsilon^2 \left((\Delta W_{t_n})^2 - \Delta_{t_n}\right).$$

In an elementary way we can eliminate the implicitness

$$X_{n+1} = \frac{X_n + \kappa\theta\Delta_{t_n} + \epsilon X_n^{\frac{1}{2}} \Delta W_{t_n} + \frac{1}{4}\epsilon^2 \left((\Delta W_{t_n})^2 - \Delta_{t_n}\right)}{1 + \kappa\Delta_{t_n}} = \frac{N(X_n)}{D(X_n)}.$$

Now we only have to verify that the numerator $N(X_n)$ is positive. Employing the idea of dominating we can accumulate all random terms in a function g:

$$\begin{aligned}
& g(\Delta W_{t_n}) && = \epsilon\sqrt{X_n}\Delta W_{t_n} + \frac{1}{4}\epsilon^2(\Delta W_{t_n})^2 \\
\Rightarrow \quad & g'(\Delta W_{t_n}) && = \epsilon\sqrt{X_n} + \frac{1}{2}\epsilon^2 \Delta W_{t_n} \\
\Rightarrow \quad & \min_{\Delta W_{t_n} \in \mathbb{R}} g(\Delta W_{t_n}) && = g\left(-\frac{2\sqrt{X_n}}{\epsilon}\right) = -X_n.
\end{aligned}$$

Knowing this lower bound enables us to exchange all random terms in the numerator

$$\begin{aligned}
N(X_n) &= X_n + \left(\kappa\theta - \frac{1}{4}\epsilon^2\right)\Delta_{t_n} + g(\Delta W_{t_n}) \\
&\geq X_n + \left(\kappa\theta - \frac{1}{4}\epsilon^2\right)\Delta_{t_n} + \min_{\Delta W_{t_n} \in \mathbb{R}} g(\Delta W_{t_n}) \\
&= X_n + \left(\kappa\theta - \frac{1}{4}\epsilon^2\right)\Delta_{t_n} - X_n \\
&= \left(\kappa\theta - \frac{1}{4}\epsilon^2\right)\Delta_{t_n} > 0,
\end{aligned}$$

since the analytical positivity causes $\kappa\theta > \epsilon^2/2$. □

This result is rather surprising as the Milstein method provides numerical positivity for arbitrary step sizes in a natural way. Now we can deal with a more general case,

$$dX_t = \kappa(\theta - X_t)dt + \epsilon X_t^p dW, \tag{18}$$

where $\frac{1}{2} < p \leq 1$. The Milstein method can also preserve positivity in this case.

Lemma 3.7 *The implicit Milstein method has an eternal life time for the integration of* (18) *if*

$$\Delta_{t_n} < \frac{1}{\epsilon^2}.$$

Proof: See [8]. □

Using the implicit Milstein method to provide positivity by dominating ΔW_{t_n} leads to an eternal life time for a wider range of mean-reverting processes.

3.2 Forward rates processes and explicit Milstein schemes

The idea of dominating leads to a general result for the explicit Milstein method, which will be applied to forward rate models later on.

Theorem 3.8 *The explicit Milstein method (15,16) for the Itô diffusion (3) has an eternal life time if the following properties hold:*

$$\begin{aligned} b(x)b'(x) &> 0, && (19)\\ x-\bar{b} &> \frac{b(x)}{2b'(x),} && (20)\\ \Delta_{t_n} &< \frac{2(x-\bar{b})b'(x)-b(x)}{\big(b(x)b'(x)-2a(x)\big)b'(x)}. && (21) \end{aligned}$$

The last condition is only necessary if the denominator is positive.

Proof: Let $x = X_n > \bar{b}$ and define $g(z) := b(x)z + \frac{1}{2}b(x)b'(x)z^2$. Then (15,16) can be written as

$$X_{n+1} = x + \left(a(x) - \frac{1}{2}b(x)b'(x)\right)\Delta_{t_n} + g(\Delta W_{t_n}).$$

According to (19) g possesses a global minimum. For that purpose an obvious calculation shows that

$$g'(z) = b(x) + b(x)b'(x)z.$$

Hence we get

$$\tilde{z} = -\frac{1}{b'(x)} \text{ with } g(\tilde{z}) = -\frac{b(x)}{2b'(x)}.$$

For this reason we can calculate the lower bound for all random terms ΔW_{t_n}. This enables us to exchange the value of $g(\Delta W_{t_n})$ by its minimum

$$X_{n+1} \geq x + \left(a(x) - \frac{1}{2}b(x)b'(x)\right)\Delta_{t_n} - \frac{b(x)}{2b'(x)}.$$

Considering the requirements (20) and (21) we get that $X_{n+1} > \bar{b}$. □

The integration in the extended Libor market model also requires the integration of the displaced diffusion (8) and constant elasticity of variance model (9). Here as well the Milstein method leads to surprising results. Note that in both cases implicit and explicit Milstein schemes coincide due to the vanishing drift term.

Integration of Displaced Diffusion

The displaced diffusion models the forward rates as

$$dX_t = \sigma_{DD}(t)(X_t + m)dW \tag{22}$$

with $\sigma_{DD}(t) := \sigma(t)\sqrt{V_t}$. Keeping the analytical behavior in mind we know that this process is bounded from below by $-m$.

Proposition 3.9 *The explicit Milstein method has an eternal life time for the integration of equation* (22)*, if* $\Delta_{t_n} < 1/\sigma_{DD}^2$.

Proof: It is sufficient to verify the properties (19–20) and the step size restriction (21) of Theorem 3.8 for $a(x) = 0$, $b(x) = \sigma_{DD}(x + m)$, $b'(x) = \sigma_{DD}$ and $\bar{b} = -m$:

$$\begin{aligned} b(x)b'(x) &= \sigma_{DD}^2(x+m) > 0, \\ x - \bar{b} - \frac{b(x)}{2b'(x)} &= (x - \bar{b}) - \frac{x+m}{2} = \frac{x+m}{2} > 0. \end{aligned}$$

The step size restriction immediately follows from $(x - \bar{b})b'(x) = b(x)$. □

Integration of Constant Elasticity of Variance

The model of constant elasticity of variance describes the forward rates as

$$dX_t = \sigma_{CEV}(t)X_t^{\alpha}dW \tag{23}$$

with $\sigma_{CEV}(t) := \sigma(t)\sqrt{V_t}$. In this case the stochastic process takes only positive values. But the boundary 0 has a different behavior with respect to α.

Proposition 3.10 *The Milstein method has the following properties for the integration of equation* (23)*:*

- $\alpha \leq \frac{1}{2} \Rightarrow$ *finite life time,*
- $\alpha > \frac{1}{2} \Rightarrow$ *eternal life time if step size is adapted.*

Proof: See [8]. □

4 Numerical tests

In the last section the theoretical results above are underlined by numerical tests. Special attention is paid to the aspect of numerical positivity or boundedness from below in the case of displaced diffusion. The tests are mainly done with regard to two different aspects. On the one hand we study whether the Monte-Carlo approximation converges to the analytical values of caplet prices which are given by closed form solutions. On the other hand we analyze convergence speed and positivity preserving properties of numerical integration schemes when applied to mean-reverting processes in order to model volatility in the Libor market model.

4.1 Pricing caplets using forward rate models

The numerical simulation of caplet prices is an important indicator to study the applicability of different integration schemes in the Libor market model. The caplet price approximation is connected with the weak convergence order as we compare expectations.

To make this point clear, let $F_k(t)$ be the exact solution at time point t for a forward rate that sets at T_k and pays at T_{k+1} and its numerical approximation $F_k^N(t)$. We assume forward rates to be defined by the Itô diffusion (1), where we assume for the moment a constant volatility $\sigma_k := \sigma(t)\sqrt{V_t}$. With $\delta_k := T_{k+1} - T_k$, the approximation error is given by

$$\left| \underbrace{\delta_k B(t,T_{k+1}) \mathrm{E}\left[(F_k(T_k)-K)^+\right]}_{C(t)} - \delta_k B(t,T_{k+1}) \mathrm{E}\left[(F_k^N(T_k)-K)^+\right] \right|$$

with the exact caplet price $C(t)$ and the price $B(t,T_{k+1})$ at time t for a zero-coupon bond expiring at time point $T_{k+1} > t$.

The numerical tests are based on three different integration schemes with constant step size $\Delta := \Delta_{t_n}$:

1. Euler: $F_{n+1}^N = F_n^N + \sigma_k \varphi(F_n^N)\Delta W$;
2. Milstein: $F_{n+1}^N = F_n^N + \sigma_k \varphi(F_n^N)\left(\Delta W + \frac{1}{2}\sigma_k \varphi'(F_n^N)\left((\Delta W)^2 - \Delta\right)\right)$;
3. log-Euler: $F_{n+1}^N = F_n^N \exp\left(-\frac{1}{2}\left(\sigma_k \frac{\varphi(F_n^N)}{F_n^N}\right)^2 \Delta + \frac{\varphi(F_n^N)}{F_n^N}\Delta W\right)$.

Provided that the analytical solution is positive, the log-Euler scheme can be derived by transforming the original stochastic differential equation into a log-normal process. By construction, positivity is preserved numerically. Of course, the log-Euler scheme is not suitable for non-positive processes.

Constant Elasticity of Variance (CEV)

At first, we analyse the numerical behavior in the CEV extension. This model is characterized by the fact that the forward rates take only nonnegative values. Indeed the boundary 0 is attainable if $0 < \alpha < 1/2$.

In Fig. 1 (left-hand side) the approximation error measured over all strikes and all maturity times is plotted against the computation time in double logarithmic scale: the Milstein scheme yields the most accurate result for a fixed computation time and, vice versa, needs the shortest computation time to achieve a required error tolerance. Particularly for strikes which are not at-the-money the Milstein scheme provides the best results.

Fig. 1 (right-hand side) shows the result for a different parameter set. In this case the log-Euler scheme leads to better results than before but nevertheless the Milstein scheme is most efficient.

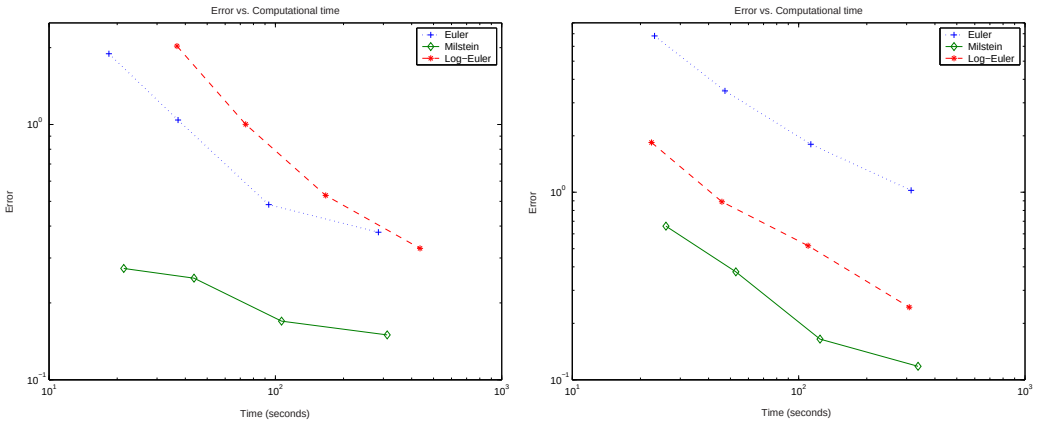

Figure 1: $dF_k(t) = \sigma_k F(t)^\alpha dW$ with:
— $F_k(0) = 0.06$, $\sigma_k = 0.04899$, $\alpha = 0.5$ (left-hand side),
— $F_k(0) = 0.06$, $\sigma_k = 0.8161$, $\alpha = 1.5$ (right-hand side);
tenor spacing $T = 0.5$, maturity times $T_k = 1, 2, 4, 6$, strikes $K = 0.04, 0.05, 0.06, 0.07, 0.08$, number of paths 2000000, Error $= ||C_{\text{exact}} - C_{MC}||_{2,K\times T_k}$ (grid points: strike $\times$ maturity times), discretization stepsizes: $\Delta_t = 1, 0.5, 0.25, 0.125$.

Displaced Diffusion

The model of displaced diffusion is a second possibility to introduce a skew in the implicit volatility surface. Indeed the problem is that the forward rates can become negative as the stochastic process takes values in the interval $[-m, \infty)$. Therefore it doesn't make sense to try to preserve positive values in the numerical integration. The analysis of the Milstein method in the chapter before shows that it provides an eternal life time with respect to the interval $[-m, \infty)$. We investigate whether the numerical approximation of caplet prices benefits from this property.

In the first example shown in Fig. 2 (left-hand side) we only present the results for the Euler and the Milstein scheme as the log-Euler method shows huge numerical instabilities. The reason for these instabilities is that the log-Euler enforces numerical positivity, which is not consistent with the analytical properties of the Itô process that is bounded from below by -0.02. Comparing the Euler and the Milstein method we obtain a familiar result. The Milstein scheme is superior with respect to the approximation and on the other hand the computational times needed for one step are nearly equal.

The negative choice of the parameter m in Fig. 2 (right-hand side) is a new challenge for the different integration schemes as the forward rates now take values in the interval $[m, \infty)$: again the Milstein method turns out to be the best.

Summing up, in both models the Milstein scheme is the method of choice: first, the computational costs are significantly lower; second, the Milstein method

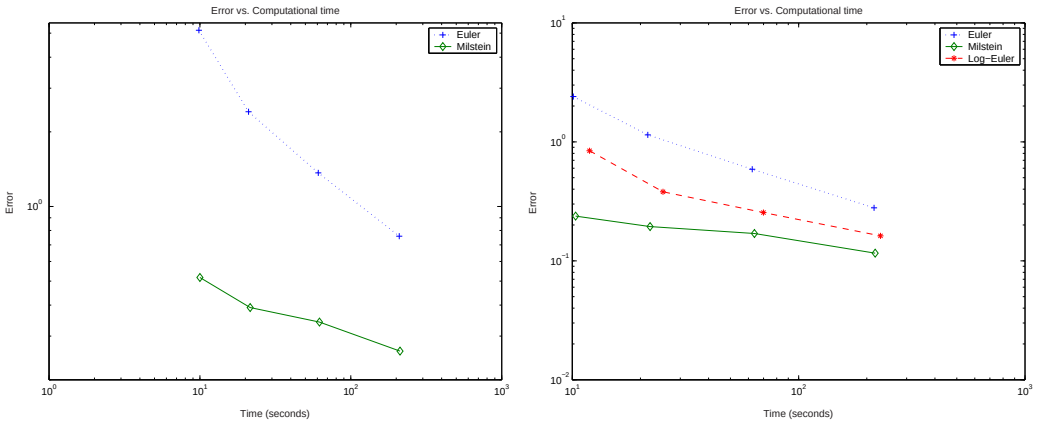

Figure 2: $dF_k(t) = \sigma_k(F(t) + m)dW$ with:
— $F_k(0) = 0.06$, $\sigma_k = 0.2$ and $m = \pm 0.02$ (left-hand/right-hand side);
tenor spacing $T = 0.5$, maturity times $T_k = 1, 2, 4, 6$, strikes $K = 0.04, 0.05, 0.06, 0.07, 0.08$, number of paths 2000000, Error $= ||C_{\text{exact}} - C_{MC}||_{2,K\times T_k}$ (grid points: strike $\times$ maturity times)

provides a better approximation to the exact caplet prices and in addition to that it preserves the analytical structure of the stochastic process. The log-Euler scheme is not suitable for displaced diffusion models with negative bounds $m > 0$.

4.2 Mean-reverting processes

Now the integration schemes are tested particularly with regard to the class of mean-reverting processes, where preservation of positivity is mandatory. Let us start with a first example:

$$dX_t = (1 - X_t)dt + \epsilon\sqrt{X_t}dW \text{ with } X_0 = 1.$$

This model is typically used to simulate stochastic volatility in the extended Libor market model [2].

The results in Table 1 clearly show that the implicit Milstein method is superior to the other methods concerning convergence speed as well as preserving positivity. As expected the Euler scheme cannot maintain this property and is less efficient than the implicit Milstein scheme due to its lower convergence order. The BIM can indeed guarantee positivity. However, the error is even bigger than in the Euler method because an extensive use of the control functions c_0 and c_1 is necessary, which lowers drastically its efficiency.

The examples make clear that the appropriate choice of an integration scheme is essential for the appropriate numerical calculation of a mean-reverting process, regarding efficiency and positivity preservation. The implicit Milstein method can

		Euler		BIM		Milstein		imp. Milstein	
Time	Stepsize	Error	Negative	Error	Negative	Error	Negative	Error	Negative
$T=1$	$\Delta_t = \frac{1}{2}$	0.2754	27.29 %	0.5187	0 %	0.2464	23.14 %	0.1503	**0 %**
	$\Delta_t = \frac{1}{4}$	0.1926	25.82 %	0.4339	0 %	0.1166	7.45 %	0.0677	**0 %**
	$\Delta_t = \frac{1}{8}$	0.1370	**21.59 %**	**0.3426**	0 %	0.0558	0.65 %	**0.0333**	**0 %**
$T=2$	$\Delta_t = \frac{1}{2}$	0.3290	45.54 %	0.7118	0 %	0.2639	34.72 %	0.2080	**0 %**
	$\Delta_t = \frac{1}{4}$	0.2241	43.39 %	0.5832	0 %	0.1269	12.12 %	0.0849	**0 %**
	$\Delta_t = \frac{1}{8}$	0.1563	**38.88 %**	**0.4499**	0 %	0.0607	1.18 %	**0.0397**	**0 %**
$T=4$	$\Delta_t = \frac{1}{2}$	0.3435	69.18 %	1.2734	0 %	0.2767	53.12 %	0.3174	**0 %**
	$\Delta_t = \frac{1}{4}$	0.2333	67.21 %	1.0188	0 %	0.1305	20.63 %	0.1019	**0 %**
	$\Delta_t = \frac{1}{8}$	0.1610	**62.44 %**	**0.7415**	0 %	0.0633	2.22 %	**0.0435**	**0 %**

Table 1: Test results for $dX_t = (1 - X_t)dt + 1.4\sqrt{X_t}dW$
Time: [0,T]; *stepsize*: Δ_t; *Error*: integration error compared with implicit Milstein ($\Delta_t = \frac{1}{4096}$); *Negative*: percentage of negative paths; choice of control functions for BIM: $c_0(x) = 1$ and $c_1(x) = 1.4x^{-\frac{1}{2}}$.

meet both requirements. Furthermore it is nearly at optimal computational costs as shown in Fig. 3. For this reason the implicit Milstein scheme is the method of choice to integrate a mean-reverting process.

5 Conclusion

In the Monte-Carlo simulation of extended Libor market models, stochastic integration schemes, which preserve the essential structures of the underlying processes, are required. Among those structural properties are, first, positivity for mean-reverting processes, which are used to model the stochastic volatility, and for constant elasticity of variance processes that can be used to model forward rates. Second, boundedness from below by a constant is a structural property when using a displaced diffusion approach in forward rate models.

These structure preserving methods have to be based on the concept of balancing or dominating the Wiener increments. We have shown that Milstein-type schemes turn out to be the method-of-choice: they combine numericall efficiency with preserving of the structural properties.

Other classes of methods, including Euler and log-Euler schemes, fail to support this structural constraints on the numerical solution. Balanced implicit methods can maintain positivity, but only at the cost of losing convergence of the methods.

Future work has to concentrate on more general stochastic differential equations that may be characterized by more general analytical structures such as

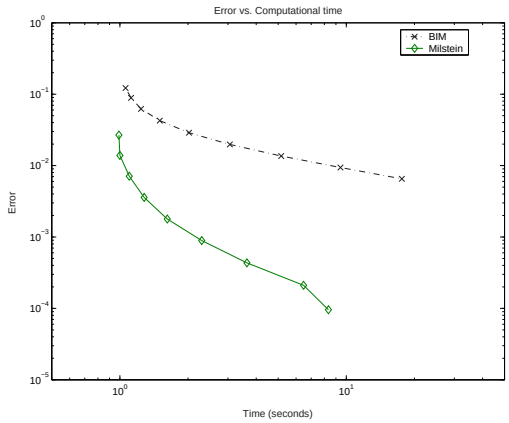

Figure 3: $dX_t = (1 - X_t)dt + 1.4\sqrt{X_t}dW$, path wise error,100000 paths, BIM: $c_0 = 1$ and $c_1(x) = 1.4x^{-\frac{1}{2}}$

equality and inequality constraints.

For this class of problems, linear-implicit schemes have been proven to be favorable in the deterministic case and are a promising approach to obtain excellent stability properties without the need to solve nonlinear systems.

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